

Tensor Product Spline Spaces.

Let π_n be a partition $a = x_0 < x_1 < \dots < x_n = b$ of an interval $[a, b]$. Then

$$S_d^r(\pi_n) = \{s \in C^r[a, b] : s|_{[x_{i-1}, x_i]} \in P_d\}$$

(where P_d is the space of polynomials of degree d) is the univariate spline space of all r times differentiable functions that can be represented as a polynomial of degree d on each subinterval of the underlying partition.

Let S_1 , S_2 , and S_3 be three such spline spaces, where the values of r_i , d_i , and n_i may be different for each S_i . Then the basic tensor product space analyzed by the software is either the bivariate space $T_2 = S_1 \oplus S_2$ or the trivariate space $T_3 = S_1 \oplus S_2 \oplus S_3$. (Actually, the software treats T_2 as the special case of T_3 where $n_3 = 1$ and $r_3 = d_3 = 0$.)

It's obvious that

$$\dim S_d^r(\pi_n) = d + 1 + (n - 1)(d - r) = r + 1 + n(d - r)$$

and

$$\dim T_3 = \prod_{i=1}^3 [r_i + 1 + n_i(d_i - r_i)]$$

and it is easy to construct minimal determining sets for these spaces.

The tp software lets you examine subspaces of T_3 that are obtained by forcing splines in the subspace to be r_i times differentiable across rectangular faces perpendicular to x_i , in the process effectively eliminating those faces.

Running the Software

The most common way to run the software is to give the command

java tp

This will ordinarily let you continue where you last quit and saved. It is possible to store a state of investigation in a file, say the file *discovery*. To continue work on the state give the command

java tp discovery

tp stores the state of an investigation in a file with the extension *.tp*. When you click on the *quit and save* button the software saves the current state in the file *tpkeepit.tp*. When you start it up the next time without specifying a different file name it will read that file and start from there. Specify just the file name, not the extension.

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If the old state cannot be restored (because the specified file does not exist, or because you have a new version of `tp` that is inconsistent with older `.tp` files) the software starts from a default state⁻²⁻. If you invoke the software from the applet it always starts in that state.

For complicated problems you may want to cause your system to dedicate additional memory to your job, by giving a command like

```
java -ms1024m -mx1024m tp
```

If you are running java on a PC under Windows, it may be necessary to tell java explicitly where to look for the class files, as in

```
java -cp . tp
```

To examine a particular spline space you want to follow these steps in this sequence:

1. Choose the dimension of the domain (bi or trivariate).
2. Chose the partition in each variable. The left end point of the interval in each case is 0, without loss of generality. The x_i must be integer. Usually they are evenly spaced, but you can change that.
3. Decide which rectangular faces (in the 3D case) or edges (in the 2D) should be effectively absent. The software simulates their removal by setting the degree of smoothness across them equal to the polynomial degree in the appropriate direction.
4. Choose the values of r and d . These may be different for each variable.
5. The splines are thought of as being represented in tensor produce Bernstein Bézier form and several drawing windows display their domain points. You click on those points to include them in a minimal determining set.

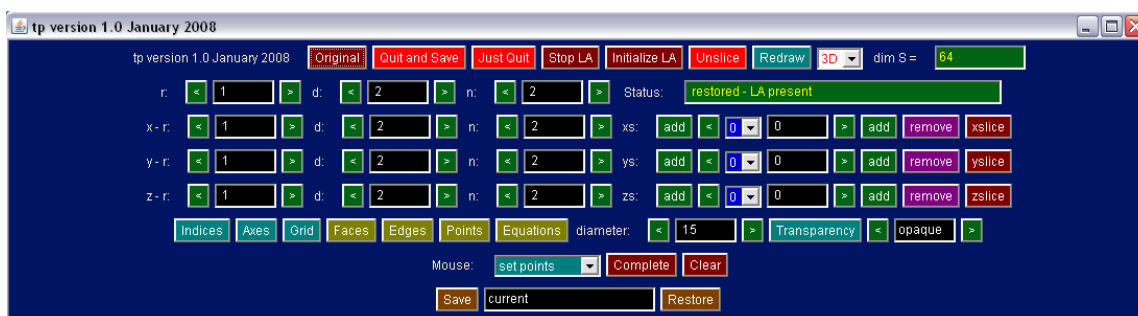


Figure 1. The Control Panel.

The Control Panel

Figure 1 shows a (reduced size) image of the Control Panel. You'll be better off running the software and looking at your actual control panel, trying out things as you read this guide, but I'll use the Figure for reference. It shows the full version of the control panel as it used for the examination of a trivariate space using the software directly (rather than

⁻²⁻ i.e., $n_i = d_i = 2, r_i = 1$.

through a browser). If you are examining a bivariate space the fifth row (marked with $z-r$: on the left), and some of the controls are absent. If you invoke the software through a browser⁻³⁻ the last line is missing (and you won't be able to save your work). For better clarity, the control panel also has a different color when you examine bivariate spaces.

Switching between bi- and trivariate

The menu in the top row of the control panel (showing **3D** in Figure 1) lets you switch back and forth between bi and trivariate spline spaces. A few settings are preserved when switching, but basically the first thing you want to do is to set that menu to **2D** or **3D** as required.

Quitting and Saving

The button “Quit and Save” in the top row attempts to save your work and closes all windows. The button “Just Quit” quits without saving. At any stage you can specify a file name in the text field in the last row and click on “Save” in that row to store your work. Similarly, by choosing the appropriate file name, and clicking “Restore” you can continue where you left off earlier.

For the saving and restoring feature to work you need to be connected to the directory containing the software, and you need to have write access in that directory. Some systems may let you specify other directories as part of the filename.

Saving files have the extension “.tp”. So if you specify “current” (the default) as the file name, the saving file will be “current.tp”. The saving files are ordinary text files containing integers and boolean values. Most, but not all, settings are saved, as are the current state of the underlying linear algebra, and the current choices of the minimal determining set.

If you want to restore the default state click on the button labeled “Original” in the top row.

Current Status

The dimension text field in the top row shows the dimension of the current spline space (if known), and the large status field in the second row gives additional information. Usually it shows the number of domain points, the number of points that have been assigned (i.e., included in the growing MDS), the number of points that have been implied by previously made assignments, and the number of points that still need to be assigned.

Choosing the partition

The underlying partition is determined by the values of r , d , and n , and the set of knots, in each of the variables x , and y , (and z in the trivariate case).

If these values are the same for all (two or three) variables you can change them using the items in the second row of the control panel. The text fields and associated switches work as they do for the 2D and 3D MDS software.

⁻³⁻ by clicking on the applet at www.math.utah.edu/~pa/tp

The next two (or three) rows let you make changes individually for each of the variables. You can choose individual values of r , d , and n as usual. The obvious restrictions apply:

$$0 \leq r \leq d \quad \text{and} \quad n \geq 1.$$

The following explanation is in terms of the variable x . Similar statements apply to the variable being y or z . By default, $x_i = i$, $i = 0, \dots, n$. The dimension of the full tensor product space is independent of the spacing. However, you can change the spacing with the remaining items at the end of each of rows 3, 4, and 5. The menu selects i , and the text field shows the value of x_i . The green buttons let you scroll through the list, and you can also choose an x_i by using the menu next to the text field. The “add” buttons add a knot to the left or right of the currently displayed x_i . “Remove” removes x_i from the partition. Entering a numerical value in the text field changes the value of x_i . In all cases, all other values are adjusted appropriately, and the left point is always zero. The “slice” buttons are present only in the 3D case. Clicking on them displays a slice showing the bivariate spline space at the appropriate value of x_i .

The Space Window

Figure 2 shows the default (3D) space drawing for reference. Of course you want to experiment with your own as you read this.

This shows a cube. Each partition has two subintervals, and the polynomial degree is $d = 2$ in all three directions. The value of smoothness is $r = 1$ in each case, but this is not indicated by the drawing. Solid black lines indicate the partition, thin blue lines indicated the Bernstein Bézier grid, and colored circles indicate the domain points.

You can see the three dimensional structure of the picture by rotating the cube. To do so drag the mouse with the right button depressed. (Alternatively, you can hold down the Control key and drag with the left mouse button.)

There are a total of 125 control points and the dimension of the full tensor product space is 64. The colors of the control points indicate their location with the respect to one of the 8 small cubes. Blue means vertex of such a cube, cyan means interior of an edge, green means interior of a face, and yellow means interior of a cube. If a domain point is chosen to be in the MDS it is drawn as a red square box. If it is implied by points already included in the MDS it is not drawn at all.

To include a point in the MDS, or remove it from the MDS, just click on it with the left mouse button. The first time you do this the underlying linear algebra will be initialized. This may take a significant amount of time, depending on the complexity of the underlying spline space.

2D Slices

2D slices of the three dimensional domain look similar, and act similarly. Indeed, in the 2D case there is only one available drawing window, corresponding to the slice $z = 0$.

Figure 3 shows a slice obtained in the 2D mode. In this case, $r = 1$ and $d = 3$ in both directions. There are a total of 256 domain points, and the dimension of the spline space is 96. There are a total of 25 squares (5 subintervals in each direction). 8 edges have been

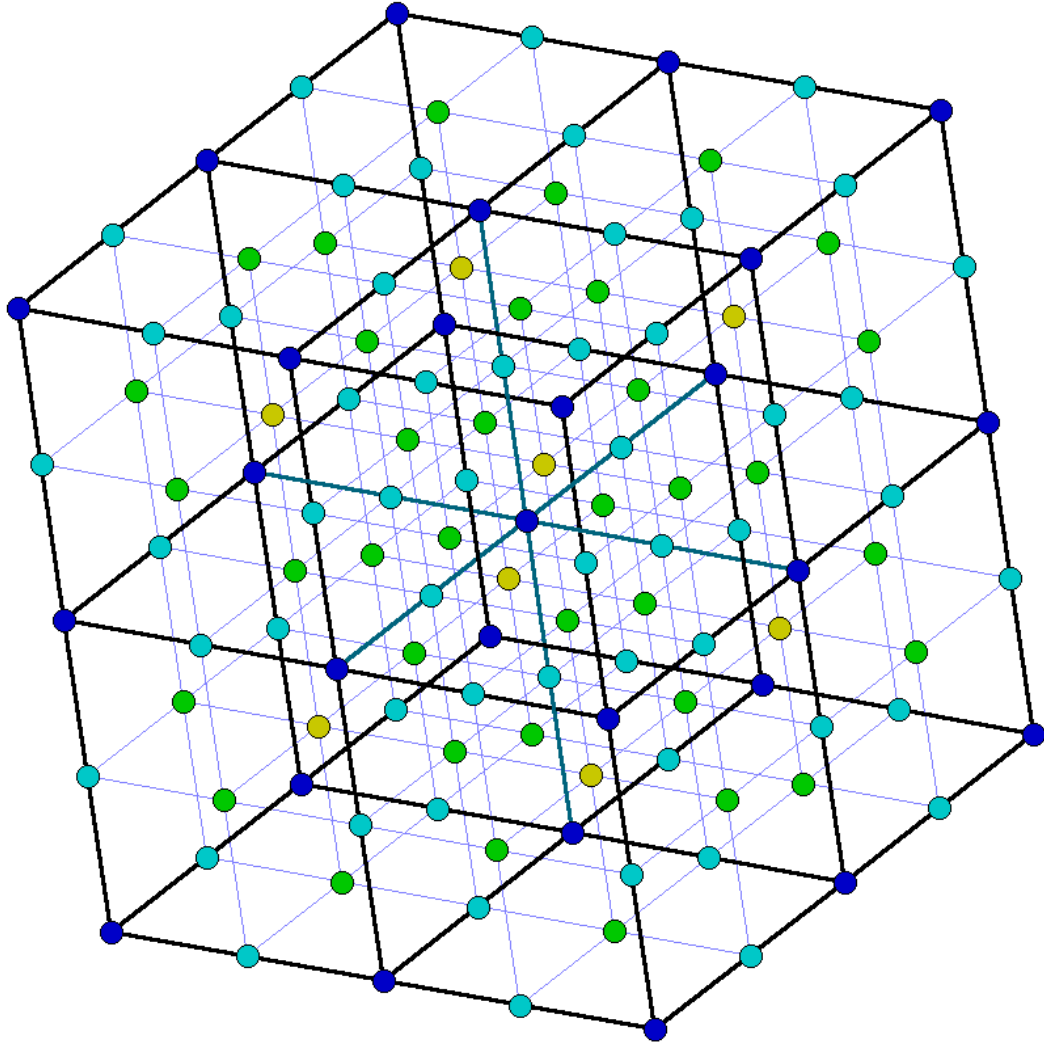


Figure 2. The Space Drawing.

removed to form a two dimensional ring. Thus there a total of 17 regions, and the spline is a polynomial (of degree 5 in each variable) on each of those regions. The regions are colored so that neighboring regions have different colors. (Two regions are neighbors if they share at least one point.) The red squares indicate the default minimal determining set (for $d = 5$ and $r = 1$). (To obtain that MDS, form the partition as indicated, and click on “complete” in the Control Panel.) All other domain points are implied, and are not shown. Clicking on any of the red points will make that point become round and assume its original color, and it will cause several other round points to reappear as well.

If you are in 3D mode, the background color of the slice indicates its direction. The direction and level are also stated in the title of the slice.

Removing Faces or Edges

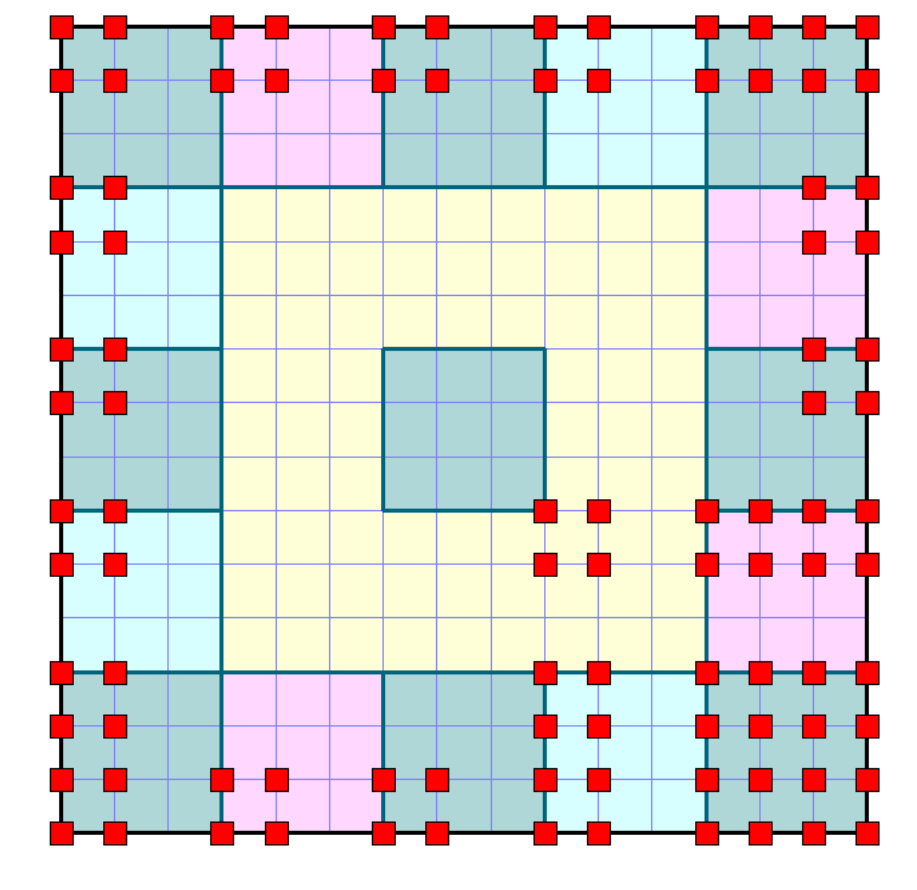


Figure 3. A slice.

You can remove faces in 3D mode, or edges in 2D mode. To do so, select “remove faces” (or “remove edges”) in the menu in the next to last row of the Control panel, and then click on a domain point in the interior of the face or edge. Clicking again undoes your choice. Clicking on a vertex in a slice removes or restores all edges connected to that vertex. If the 2D window corresponds to a slice of a 3D domain then a face in that domain is removed or restored.

Miscellaneous Commands

Following is a list of items available on the control panel, including all those that have not yet been mentioned. The list is ordered alphabetically.

Axes Toggles on and off the drawing of coordinate axes. Note that by java conventions the origin is in the upper left corner of a drawing.

Check LA Computes and displays information about the current state of the Linear Algebra, specifically the dimensions of the current matrix and the percentage of non-zero entries. It also checks that all rows add to zero, as they must. (If they don’t please let me know.)

Clear Remove all points so far chosen from the current minimal determining set.

- Complete** Show the current internally maintained minimal determining set.
- Diameter** is the diameter of the domain points measured in pixels. It can be adjusted up or down as necessary.
- Edges** causes a similar report for edges, and
- Equations** toggles on or off the printing of the underlying equations to standard output during the initialization of the linear algebra.
- Faces** causes report on existing faces and their properties to standard output.
- Grid** toggles drawing of the B-form grid.
- Indices** toggles drawing the internally used indices of the domain points. This maybe useful for noting correspondences between slices and the 3D window.
- Initialize LA** Starts the preprocessing. This is also caused, if necessary, by other actions, such as clicking on a domain point in the set coefficient mode (see below).
- Original** Restores the default state, identical to the starting date for the web version.
- Points** prints a list of domain points and their properties to standard output.
- Redraw** causes all windows to be redrawn. This should never be necessary. You can also cause any particular drawing window to be redrawn by focusing on it and pressing any key on the keyboard.
- set points** If selected in the menu in the bottom row, will cause the selected point to be added to, or removed from, the current minimal determining set.
- set facets** If selected in the menu in the bottom row, will cause all points to be processed that lie in the same edge or face as the selected point.
- Stop LA** Stops the initial processing of the linear system (which may be time consuming.)
- Transparency** Faces that are removed in the 3D mode are indicated by color according to their direction. These color can be chosen to be transparent, and their opaqueness can be adjusted.
- Unslice** makes all slices invisible in the 3D mode.