Smooth Macro-Elements Based on Powell-Sabin Triangle Splits

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Abstract. Macro-elements of smoothness C^r on Powell-Sabin triangle splits are constructed for all $r \geq 0$. These new elements are improvements on elements constructed in [13] in that certain unneeded degrees of freedom have been removed.

§1. Introduction

A bivariate macro-element defined on a triangle T consists of a finite dimensional linear space $\mathcal{S}$ defined on T, and a set Λ of linear functionals forming a basis for the dual of $\mathcal{S}$.

It is common to choose the space $\mathcal{S}$ to be a space of polynomials or a space of piecewise polynomials defined on some subtriangulation of T. The members of Λ, called degrees of freedom, are usually taken to be point evaluations of derivatives.

A macro-element defines a local interpolation scheme. In particular, if f is a sufficiently smooth function, then we can define the corresponding interpolant as the unique function $s \in \mathcal{S}$ such that $\lambda s = \lambda f$ for all $\lambda \in \Lambda$. We say that a macro-element has smoothness C^r provided that if the element is used to construct an interpolating function locally on each triangle of a triangulation $\triangle$, then the resulting piecewise function is C^r continuous globally.

The first C^r macro-elements were constructed using polynomials of degree $4r+1$, see Remark 8.1. To get macro-elements using lower degree polynomials, it is necessary to split the triangle. Here we focus on the case where T is split into six subtriangles as follows. Suppose the vertices of T are $\mathcal{V} := \{v_1, v_2, v_3\}$, that v_T is a point inside of T, and that $\mathcal{W} := \{w_1, w_2, w_3\}$ are points on the edges $\langle v_i, v_{i+1}\rangle$. Then the corresponding Powell-Sabin split T_{PS} of T consists of the six triangles

$$T^{[i]} := \langle v_T, v_i, w_i\rangle, \quad \widetilde{T}^{[i]} := \langle v_T, w_i, v_{i+1}\rangle, \qquad i = 1, 2, 3, \tag{1.1}$$

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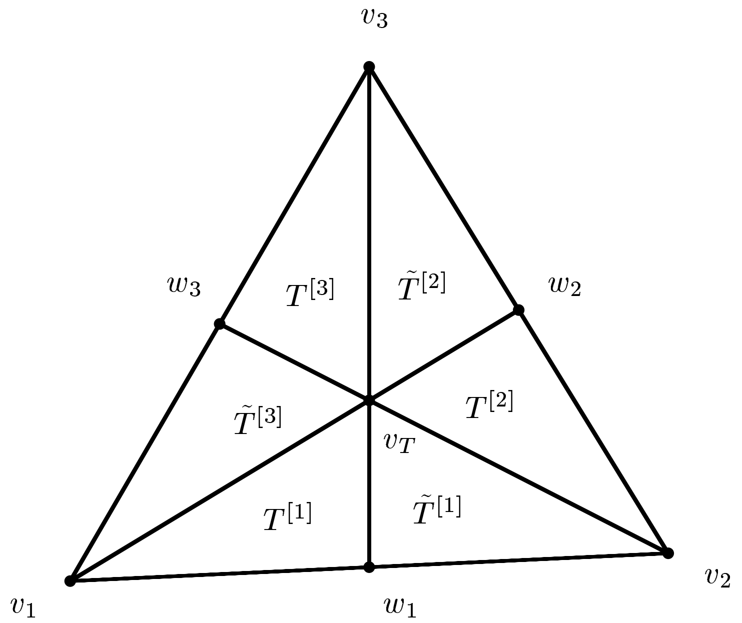

Fig. 1. The Powell-Sabin split.

where we identify $v_4 = v_1$, see Fig. 1. Let $\mathcal{E}$ be the set of edges $\langle v_T, w_i \rangle$ for $i = 1, 2, 3$.

The classical Powell-Sabin macro element [17] is based on the triangulation T_{PS} and the 9-dimensional space of C^1 quadratic splines on T_{PS}. The 9 degrees of freedom are chosen to be the values and gradients at the three vertices of T.

Several authors have created smoother versions of this classical quadratic PS-element, see [5,13,18–21] and references therein. C^r elements based on the PS-split and using polynomials of the lowest possible degree were constructed recently in [13] based on the superspline spaces

$$\begin{aligned} \widetilde{\mathcal{S}}_d^{r,\rho,\mu}(T_{PS}) := \{ s \in C^r(T) : \; & s \text{ is a piecewise polynomial of degree } d \text{ on } T_{PS}, \\ & s \in C^\rho(v) \text{ for all } v \in \mathcal{V}, s \in C^\mu(v) \text{ for all } v \in \mathcal{W}, \\ & \text{and } s \text{ is } C^\mu \text{ across all edges in } \mathcal{E} \} \end{aligned} \tag{1.2}$$

for appropriate values of ρ and μ depending on r. As usual, $C^\mu(v)$ means that all polynomials on triangles sharing the vertex v have common derivatives up to order μ at that vertex. Note that (1.2) is a superspline space with additional continuity across certain interior edges.

The purpose of this paper is to show how the elements of [13] can be improved by reducing the number of degrees of freedom, resulting in macro-elements on the PS-split which can be parametrized in terms of point-evaluations of derivatives at the vertices and certain cross-boundary derivatives only.

The paper is organized as follows. In Sect. 2 we introduce some notation, and in Sect. 3 we review the theory of minimal determining sets for spline spaces. In

Sect. 4 we state and prove the main result of the paper. In Sect. 5 we illustrate the result with some examples, and in Sect. 6 we translate our results to nodal form. in Sect. 7 we discuss the use of our elements for Hermite Interpolation. The paper concludes with remarks in Sect. 8.

§2. Preliminaries

Our starting point is [13], and we closely follow the notation used there and in the papers [1–14]. Given a triangle $T := \langle u_1, u_2, u_3\rangle$ and an integer d, we use the Bernstein-Bézier representation

$$p = \sum_{i+j+k=d} c^T_{ijk} B^d_{ijk},$$

for polynomials of degree d. Here B^d_{ijk} are the Bernstein polynomials of degree d associated with T defined by

$$B^d_{ijk}(u) := \frac{d\,!}{i\,!\,j\,!\,k\,!}\alpha^i\beta^j\gamma^k, \qquad i+j+k=d, \tag{2.1}$$

where (α,β,γ) are the barycentric coordinates of the point $u \in \mathbb{R}^2$ in terms of the triangle T.

As usual, we denote the associated domain points by

$$\xi^T_{ijk} := \frac{(iu_1 + ju_2 + ku_3)}{d}, \qquad i+j+k=d.$$

We will work with rings and disks of domain points defined by

$$R^T_n(u_1) := \{\xi^T_{ijk} :\ i = d-n\},$$
$$D^T_n(u_1) := \{\xi^T_{ijk} :\ i \ge d-n\},$$

with similar definitions at the other vertices of T. In the sequel we shall frequently say that a coefficient is on a ring or in a disk when what we really mean is that its associated domain point is in that location.

Suppose that $T := \langle u_1, u_2, u_3\rangle$ and $\widetilde{T} := \langle u_4, u_3, u_2\rangle$ are two adjoining triangles which share the edge $e := \langle u_2, u_3\rangle$. Let p and $\tilde{p}$ be two polynomials of degree d with B-coefficients c_{ijk} and $\tilde{c}_{ijk}$ relative to T and $\widetilde{T}$, respectively. Then it is well known that p and $\tilde{p}$ join with C^r continuity across the edge e if and only if

$$\tilde{c}_{n,m-n,d-m} = \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(u_4), \tag{2.2}$$

for $m = n,\dots,d$ and $n = 0,\dots,r$. Here B^n_{ijk} are the Bernstein polynomials of degree n on the triangle T.

Assuming that the coefficients of p are known and that $\tilde{p}$ joins p with C^r continuity, the smoothness conditions (2.2) can be used to compute the coefficients $\tilde{c}_{n,m-n,d-m}$ of $\tilde{p}$ for $0 \le n \le r$. They can also be used in situations where some of the coefficients of both p and $\tilde{p}$ are known and others are unknown. We need the following lemma which shows how this works for computing coefficients on the ring $R^T_m(u_2) \cup R^{\widetilde{T}}_m(u_2)$, assuming that an appropriate set of smoothness conditions across the edge e are satisfied.

Lemma 2.1. [2] *Suppose T and $\widetilde{T}$ are as above, and that all coefficients c_{ijk} and $\tilde{c}_{ijk}$ of the polynomials p and $\tilde{p}$ are known except for*

$$\begin{aligned} c_\nu &:= c_{\nu,d-m,m-\nu}, \qquad && \nu = \ell+1,\ldots,q, \\ \tilde{c}_\nu &:= \tilde{c}_{\nu,m-\nu,d-m}, \qquad && \nu = \ell+1,\ldots,\tilde{q}, \end{aligned} \tag{2.3}$$

for some $\ell, m, q, \tilde{q}$ with $0 \le q, \tilde{q}$, $-1 \le \ell \le q, \tilde{q}$, and $q+\tilde{q}-\ell \le m \le d$. Then these coefficients are uniquely determined by the smoothness conditions

$$\tilde{c}_{n,m-n,d-m} = \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(u_4), \qquad \ell+1 \le n \le q+\tilde{q}-\ell. \tag{2.4}$$

§3. Minimal determining sets

Let $\mathcal{S}^0_d(\triangle)$ be the space of continuous splines of degree d on a triangulation $\triangle$, and let $\mathcal{D}_{d,\triangle}$ be the union of the sets of domain points associated with each triangle of $\triangle$. Then it is well known that each spline in $\mathcal{S}^0_d(\triangle)$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}}$. In particular, the coefficients of the polynomial $s|_T$ are precisely $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}\cap T}$.

In this paper we are interested in subspaces $\mathcal{S}$ of $\mathcal{S}^0_d(\triangle)$ which satisfy additional smoothness conditions, including smoothness conditions across edges, smoothness conditions at vertices, and certain special individual smoothness conditions. Suppose that $T := \langle u_1, u_2, u_3\rangle$ and $\widetilde{T} := \langle u_4, u_3, u_2\rangle$ are two adjoining triangles which share the edge $e := \langle u_2, u_3\rangle$. Let c_{ijk} and $\tilde{c}_{ijk}$ be the coefficients of the B-representations of s_T and $s_{\widetilde{T}}$, respectively. Then for any $n \le m \le d$, let

$$\tau^n_{m,e} s := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(u_4), \tag{3.1}$$

where B^n_{ijk} are the Bernstein polynomials of degree n on the triangle T. In terms of these linear functionals, the conditions (2.2) for C^r smoothness across the edge e can be restated as

$$\tau^n_{m,e} s = 0, \qquad n \le m \le d, \quad 0 \le n \le r.$$

If s is a spline in $\mathcal{S}^0_d(\triangle)$ which satisfies additional smoothness conditions, then clearly we cannot independently choose all of its coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}}$. We recall that a determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ is a subset $\mathcal{M}$ of the set of domain points $\mathcal{D}_{d,\triangle}$ such that if $s \in \mathcal{S}$ and $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $c_\xi = 0$ for all $\xi \in \mathcal{D}_{d,\triangle}$, *i.e.*,, $s \equiv 0$. The set $\mathcal{M}$ is called a minimal determining set (MDS) for $\mathcal{S}$ if there is no smaller determining set. It is known that $\mathcal{M}$ is a MDS for $\mathcal{S}$ if and only if every spline $s \in \mathcal{S}$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$.

A MDS $\mathcal{M}$ is called stable provided that for each $\theta > 0$ there exist positive constants K_1 and K_2 depending only on θ such that $K_1\|c\|_\infty \le \|s\|_\infty \le K_2\|c\|_\infty$, whenever $\triangle$ is a triangulation whose smallest angle exceeds θ. Here $\|c\| := \max_{\xi\in\mathcal{M}} |c_\xi|$, and $\|s\|_\infty$ is the maximum of $|s(x)|$ over the union of the triangles of $\triangle$.

We conclude this section with some additional notation. If v is a vertex of $\triangle$, the rings $R_m(v)$ and disks $D_m(v)$ are defined to be the unions of the rings $R_m^T(v)$ and disks $D_m^T(v)$, respectively, taken over all triangles T attached to v.

§4. The main result

We begin by defining certain spline spaces defined on the PS split. The notation is somewhat complicated since (as we discovered from computational experiments, see Remark 8.2), it is necessary to distinguish eight cases based on the value of r. Let $\widetilde{\mathcal{S}}_d^{r,\rho,\mu}(T_{PS})$ be the space of supersplines defined in (1.2). For ease of notation, in the remainder of the paper we will write

$$\tau_{m,l}^n := \tau_{m,\langle v_l, v_T\rangle}^n, \qquad 1 \le l \le 3, \tag{4.1}$$

where $\tau_{m,e}^n$ are the linear functionals defined in (3.1).

Definition 4.1. *Given $r \ge 0$, let k be such that $8k \le r \le 8k+7$. Let ρ, μ, d, p, q, n be the associated integers defined in Table 1, and let $\mathcal{S}_r(T_{PS})$ be the subspace of all splines $s \in \widetilde{\mathcal{S}}_d^{r,\rho,\mu}(T_{PS})$ such that*

$$\tau_{\rho+j,1}^{r+i} s = \tau_{\rho+j,2}^{r+i} s = \tau_{\rho+j,3}^{r+i} s = 0, \qquad 1 \le i \le \begin{cases} 2j-1, & r \text{ even} \\ 2j-2, & r \text{ odd}, \end{cases} \tag{4.2}$$

for $1 \le j \le p$, and

$$\begin{aligned} \tau_{\rho+j,1}^{r+i} s &= 0, \qquad 1 \le i \le n-2j, \\ \tau_{\rho+j,2}^{r+i} s &= 0, \qquad 1 \le i \le n-2j-1, \\ \tau_{\rho+j,3}^{r+i} s &= 0, \qquad 1 \le i \le n-2j-2, \end{aligned} \tag{4.3}$$

for $p+1 \le j \le q$.

Tab. 1. Parameters for the space $\mathcal{S}_r(T_{PS})$.

r	ρ	μ	d	p	q	n
$8k$	$12k$	$12k+1$	$18k+1$	k	$2k$	$4k+3$
$8k+1$	$12k+1$	$12k+1$	$18k+2$	k	$2k$	$4k+2$
$8k+2$	$12k+3$	$12k+3$	$18k+5$	k	$2k+1$	$4k+3$
$8k+3$	$12k+4$	$12k+5$	$18k+7$	$k+1$	$2k+1$	$4k+4$
$8k+4$	$12k+6$	$12k+7$	$18k+10$	$k+1$	$2k+1$	$4k+5$
$8k+5$	$12k+7$	$12k+7$	$18k+11$	$k+1$	$2k+1$	$4k+4$
$8k+6$	$12k+9$	$12k+9$	$18k+14$	$k+1$	$2k+2$	$4k+5$
$8k+7$	$12k+10$	$12k+11$	$18k+16$	$k+1$	$2k+2$	$4k+6$

Tab. 2. Parameters for Theorem 4.2.

r	a	b	κ
$8k$	57	3	$6k^2+3k$
$8k+1$	93	9	$6k^2-3k$
$8k+2$	165	30	$6k^2+3k+1$
$8k+3$	207	48	$6k^2+3k$
$8k+4$	279	87	$6k^2+9k+3$
$8k+5$	315	111	$6k^2+3k$
$8k+6$	387	168	$6k^2+9k+4$
$8k+7$	429	207	$6k^2+9k+3$

Theorem 4.2. *Given $8k \le r \le 8k+7$, let a, b be the associated integers in Table 2. Then*

$$\dim \mathcal{S}_r(T_{PS}) = 222k^2 + ak + b. \tag{4.4}$$

Moreover, the following set $\mathcal{M}_r$ is a stable MDS for $\mathcal{S}_r(T_{PS})$:

1) $D_\rho^{T^{[i]}}(v_i)$ *for* $i = 1,2,3$,

2) $\{\xi^{T^{[i]}}_{j,j+\mu-d-1,2d+1-\mu-2j}, \ldots, \xi^{T^{[i]}}_{j,0,d-j}\}$ *for* $j = d-\mu+1, \ldots, r$ *and* $i = 1,2,3$.

Here ρ and μ are the integers in Table 1.

Proof: First we show that $\mathcal{M}_r$ is a determining set. Suppose s is a spline in $\mathcal{S}_r(T_{PS})$ and that $c_\xi = 0$ for all $\xi \in \mathcal{M}_r$. We claim that all other coefficients must also be zero, and so $s \equiv 0$. For ease of notation, we write $R_j(v_i)$ for the ring $R_j^{T^{[i]}}(v_i)$, where $T^{[i]}$ are the triangles defined in (1.1). We also define

$$E_{j,i} := \{\xi^{T^{[i]}}_{j,d-j-\nu,\nu}\}_{\nu=0}^{d-j} \cup \{\xi^{\widetilde{T}^{[i]}}_{j,\nu,d-j-\nu}\}_{\nu=0}^{d-j}.$$

These are the domain points in the j-th row of $T^{[i]} \cup \widetilde{T}^{[i]}$ parallel to the edge $\langle v_i, v_{i+1}\rangle$.

The process proceeds in steps, and at any point in it, we refer to the coefficients which have already been shown to be zero as *known*, and refer to the remaining coefficients as *unknown*. First, we use the smoothness conditions to compute the values of the unknown coefficients in the disks $D_\rho(v_l)$ for $l = 1,2,3$. They turn out to be zero since they depend directly on the values of c_ξ for $\xi \in \mathcal{M}_r$ which are all assumed to be zero. We then use the C^μ smoothness conditions across the edges $\langle v, w_l\rangle$ and Lemma 2.1 to compute the μ unknown coefficients in each of the rows $E_{0,l}, \ldots, E_{r,l}$, $l = 1,2,3$. This involves nonsingular systems with zero right-hand sides. Then for each $1 \le j \le p$ and $l = 1,2,3$, we compute the $2(\rho - r + j - 1) + 1$ unknown coefficients on the ring $R_{\rho+j}(v_l)$. The computation of the remaining unknown coefficients divides into six cases.

Case 1: $r = 8k, 8k+1$. We perform a cycle of computations. For each $j = p+1, \ldots, q$:

a) compute the $2(\rho - r + j - 1) + 1$ unknown coefficients on the ring $R_{\rho+j}(v_1)$,
b) compute the $d - r + 2p - 2j + 1$ unknown coefficients in the rows $E_{r+2(j-p)-1,1}$ and $E_{r+2(j-p)-1,3}$,
c) compute the $2(\rho - r + j - 1)$ unknown coefficients on the ring $R_{\rho+j}(v_2)$,
d) compute the $d - r + 2p - 2j + 1$ unknown coefficients in the row $E_{r+2(j-p)-1,2}$,
e) compute the $2(\rho - r + j - 1) - 1$ unknown coefficients on the ring $R_{\rho+j}(v_3)$.
f) if $j < q$, compute the $d - r + 2p - 2j$ unknown coefficients in the rows $E_{r+2(j-p),l}$, $l = 1, 2, 3$.

At this point the only remaining unknown coefficients lie inside the disk $D_\mu(v_T)$. In view of the supersmoothness μ at the vertex v_T, we can consider these coefficients to be those of a polynomial P of degree $\mu = 12k+1$ on a single triangle $\langle u_1, u_2, u_3 \rangle$. Since we have already computed all coefficients of s in the disks $D_{\rho+q}(v_l)$, this gives us all coefficients of P in the disks $D_{8k}(u_l)$ for $l = 1, 2, 3$. These determine all coefficients of P, and hence all remaining coefficients of s.

Case 2: $r = 8k + 2$. In this case we do the cycle of computations of Case 1 for $j = p+1, \ldots, q-1$. Next we compute the $r + 1$ unknown coefficients on the ring $R_{\rho+q}(v_1)$. This gives us all of the coefficients of a polynomial P of degree $\mu = 12k+3$ in the disks $D_{8k+2}(u_1)$ and $D_{8k+1}(u_l)$ for $l = 2, 3$. These determine all coefficients of P, and hence all remaining coefficients of s.

Case 3: $r = 8k + 3$. If $k > 0$, we first compute the $d - r - 1$ unknown coefficients in each of the rows $E_{r+1,l}$ for $l = 1, 2, 3$. We then do the following cycle of computations. For each $j = p+1, \ldots, q-1$:

a) compute the $2(\rho - r + j - 1) + 1$ unknown coefficients on the ring $R_{\rho+j}(v_1)$,
b) compute the $d - r + 2p - 2j + 1$ unknown coefficients in the rows $E_{r+2(j-p),1}$ and $E_{r+2(j-p),3}$,
c) compute the $2(\rho - r + j - 1)$ unknown coefficients on the ring $R_{\rho+j}(v_2)$,
d) compute the $d - r + 2p - 2j + 1$ unknown coefficients in the row $E_{r+2(j-p),2}$,
e) compute the $2(\rho - r + j - 1) - 1$ unknown coefficients on the ring $R_{\rho+j}(v_3)$,
f) if $j < q$, compute the $d - r + 2p - 2j$ unknown coefficients in the rows $E_{r+2(j-p)+1,l}$, $l = 1, 2, 3$.

To complete the computation, we find the unknown coefficients on $R_{\rho+q}(v_1)$, on edge $E_{r+2(q-p)-1,1}$ if $q > p$, and on $R_{\rho+q}(v_2)$. This gives us all coefficients of the degree $\mu = 12k + 5$ polynomial P in the disks $D_{8k+3}(u_l)$, $l = 1, 2$, and $D_{8k+2}(u_3)$. These determine all coefficients of P, and hence all remaining coefficients of s.

Case 4: $r = 8k + 4, r = 8k + 5$. We proceed as in Case 3, except now we do the cycles for all $j = p+1, \ldots, q$. This gives us all coefficients of the degree $\mu = 12k+7$

polynomial P in the disks $D_{8k+4}(u_l)$, $l = 1, 2, 3$. These determine all coefficients of P, and hence all remaining coefficients of s.

Case 5: $r = 8k + 6$. We first compute the unknown coefficients in the rows $E_{r+1,l}$ for $l = 1, 2, 3$, and then perform the cycles as in Case 3 for $j = p + 1, \ldots, q - 1$. To complete the computation, we then compute the unknown coefficients on the ring $R_{\rho+q}(v_1)$. This gives us all coefficients of the degree $\mu = 12k + 9$ polynomial P in the disks $D_{8k+6}(v_1)$ and $D_{8k+5}(u_l)$, $l = 2, 3$. These determine all coefficients of P, and hence all remaining coefficients of s.

Case 6: $r = 8k+7$. We begin by doing the cycles of Case 1 for $j = p+1, \ldots, q-1$. Then we compute the unknown coefficients on $R_{\rho+q}(v_1)$, on edge $E_{r+2(q-p)-1,1}$, and ring $R_{\rho+q}(v_2)$. This gives us all coefficients of the degree $\mu = 12k + 11$ polynomial P in the disks $D_{8k+7}(v_1)$, $l = 1, 2$ and $D_{8k+6}(u_3)$. These determine all coefficients of P, and hence all remaining coefficients of s.

This completes the proof that $\mathcal{M}_r$ is a determining set for $\mathcal{S}_r(T_{PS})$. We now show that it is minimal. Clearly,

$$\#\mathcal{M}_r = 3\binom{\rho+2}{2} + 3\binom{r+\mu-d+1}{2}, \tag{4.5}$$

which reduces to the number listed in (4.4). Now for $r = 2m$, our spline space $\mathcal{S}_r(T_{PS})$ is a subspace of the spline spaces of Theorems 4.1, 5.1, 6.1, and 7.1 of [13]. $\mathcal{S}_r(T_{PS})$ is the subspace satisfying the smoothness conditions listed in (4.2)–(4.3). It is easy to see that the number of such conditions depends on k, and is given by the integer κ listed in Table 2. Thus, using the stated theorems, we have

$$\dim \mathcal{S}_r(T_{PS}) \geq \begin{cases} \frac{57m^2+54m+13}{4} - \kappa, & r = 2m,\ m \text{ odd}, \\ \frac{57m^2+60m+12}{4} - \kappa, & r = 2m,\ m \text{ even}, \\ \frac{57m^2+90m+36}{4} - \kappa, & r = 2m+1,\ m \text{ even}, \\ \frac{57m^2+96m+39}{4} - \kappa, & r = 2m+1,\ m \text{ odd}. \end{cases}$$

But this is equal to $\#\mathcal{M}_r$, and we conclude that $\mathcal{M}_r$ is a MDS and $\dim \mathcal{S}_r(T_{PS}) = \#\mathcal{M}_r$.

To verify the stability of $\mathcal{M}_r$, we observe that all of the above computations are stable since the determinants of the systems arising in Lemma 2.1 (cf. the proof in [2]) are bounded away from zero by a constant depending only on the smallest angle in T_{PS} and the barycentric coordinates of the center point v_T. $\square$

§5. Examples

In this section we illustrate the construction of the previous section for several values of r. For reference, in Table 3 we list the values of r, ρ, d and the corresponding dimensions for $1 \leq r \leq 15$.

Tab. 3. Properties of $\mathcal{S}_r(T_{PS})$.

r	ρ	d	dim
1	1	2	9
2	3	5	30
3	4	7	48
4	6	10	87
5	7	11	111
6	9	14	168
7	10	16	207
8	12	19	282
9	13	20	324
10	15	23	417
11	16	25	477
12	18	28	588
13	19	29	648
14	21	32	777
15	22	34	858

Figures 2 – 9 show the elements corresponding to $r = 2, \ldots, 9$. This covers each of the eight cases. In each figure we mark the domain points corresponding to item 1) of Theorem 4.2 with black dots, and those corresponding to item 2) with black squares. Each figure also includes additional information on the special smoothness conditions and the calculation of unset coefficients. In particular, we mark the endpoints of special smoothness conditions with a bracket, and also illustrate the steps in the calculation of unset coefficients by drawing lines through the groups of domain points being calculated in each step. The grey line marks the disk $D_\mu(v_T)$ corresponding to the supersmoothness $C^\mu(v_T)$. The grey-shaded area marks the union of the disks $D_\rho(v_i)$ and the sets $E_{0,i}, \ldots, E_{r,i}$ for $i = 1, 2, 3$.

We now discuss two examples in detail.

Example 5.1. *The space $\mathcal{S}_2(T_{PS})$ is the 30 dimensional subspace of $\widetilde{\mathcal{S}}_5^{2,3,3}(T_{PS})$ which satisfies one additional smoothness condition corresponding to the linear functional $\tau_{4,1}^3$.*

Discussion: Fig. 2 illustrates the MDS for this space. It consists of 10 points in each of the disks $D_3(v_i)$, marked with black dots. The tip of the smoothness condition corresponding to $\tau_{4,1}^3$ is marked with a bracket. In following the proof of Theorem 4.2, after computing the coefficients corresponding to the points in the sets $E_{0,l}, \ldots, E_{2,l}$ for $l = 1, 2, 3$, we use Lemma 2.1 to compute the remaining three coefficients on ring $R_4(v_1)$. $\square$

Example 5.2. *The space $\mathcal{S}_4(T_{PS})$ is the 87 dimensional subspace of $\widetilde{\mathcal{S}}_{10}^{4,6,7}(T_{PS})$ which satisfies three additional smoothness conditions associated with $\tau_{7,l}^5$ for $l =$ 1, 2, 3.*

Discussion: Fig. 4 shows the MDS for this element. It consists of 28 points in each of the disks $D_6(v_i)$, marked with dark dots, along with the three points marked with squares. $\square$

§6. Nodal degrees of freedom

It is common in the finite-element literature to describe the degrees of freedom of macro-elements in terms of derivatives. In this section we show that there is a natural way to do this for the macro elements introduced in Definition 4.1. Let D_x and D_y be the usual partial derivatives. Let δ_t be point evaluation at t. If $e := \langle v_l, v_{l+1}\rangle$ is one of the edges of T, we denote the derivative normal to that edge by D_e. Let

$$\eta^j_{e,i} := \frac{(j+1-i)v_l + iv_{l+1}}{j+1}, \qquad i = 1,\ldots,j. \tag{6.1}$$

Theorem 6.1. *Given $8k \le r \le 8k+7$, let $\mathcal{S}_r(\triangle_{PS})$ be the spline space defined in Definition 4.1 where ρ, μ, d are as in Table 1. Then any spline $s \in \mathcal{S}_r(\triangle_{PS})$ is uniquely determined by the following set of data:*

1) $\{\delta_{v_i} D_x^\alpha D_y^\beta\}_{0\le\alpha+\beta\le\rho}$, *for* $i = 1, 2, 3$,
2) $\{\delta_{\eta^j_{e_l,i}} D_{e_l}\}_{i=1}^j$, *for* $j = 1,\ldots,r-d+\mu$ *and* $l = 1, 2, 3$, *where* $e_l := \langle v_l, v_{l+1}\rangle$.

Proof: It is easy to see that setting this nodal data is equivalent to setting the B-coefficients listed in Theorem 4.2. See [9] for some explicit formulae. $\square$

§7. Hermite Interpolation of Scattered Data

In this section we briefly examine the use of our macro-elements for interpolation of Hermite data at a set of scattered points $\mathcal{V} := \{(x_i, y_i)\}_{i=1}^V$. Our aim is to construct a C^r spline which interpolates this data.

We begin by triangulating the data points. Let $\triangle$ be a triangulation with vertices at the points of V. For many applications, this might be the Delaunay triangulation. Let $\triangle_{PS}$ be the triangulation obtained from $\triangle$ by performing the following steps:

1) insert the incenters of each triangle T of $\triangle$,
2) connect the incenters of neighboring triangles,
3) connect the incenter of each triangle T to the three vertices of T and to the centers of any edges of T which are boundary edges of $\triangle$.

Thus, each triangle of $\triangle$ has been subjected to a PS-split. To perform interpolation, we make use of the space

$$\mathcal{S}_r(\triangle_{PS}) := \{s \in C^r(\Omega) : \ s|_T \in \mathcal{S}_r(T_{PS}) \text{ for all } T \in \triangle\}, \tag{7.1}$$

where Ω is the union of the triangles in $\triangle$.

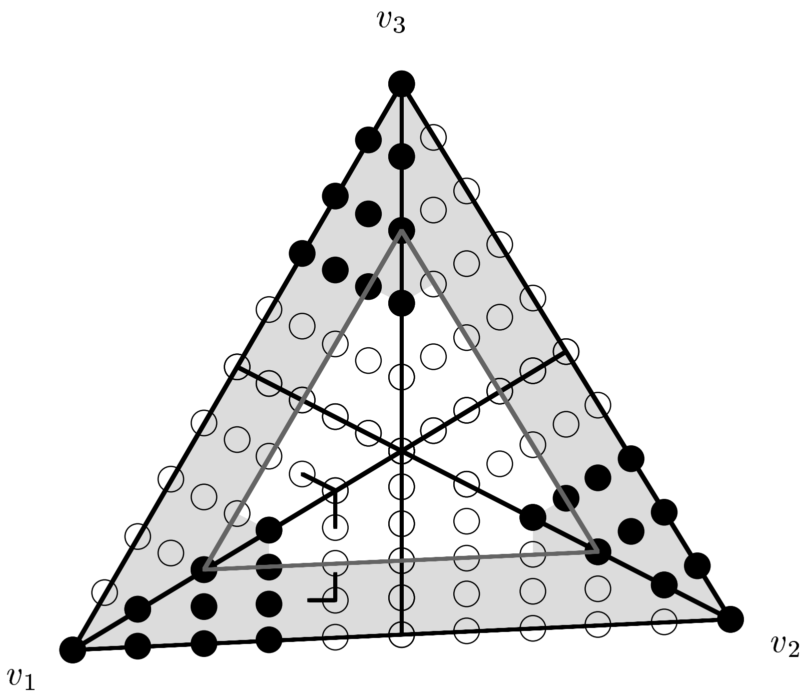

Fig. 2. The macro-element $\mathcal{S}_2(T_{PS})$.

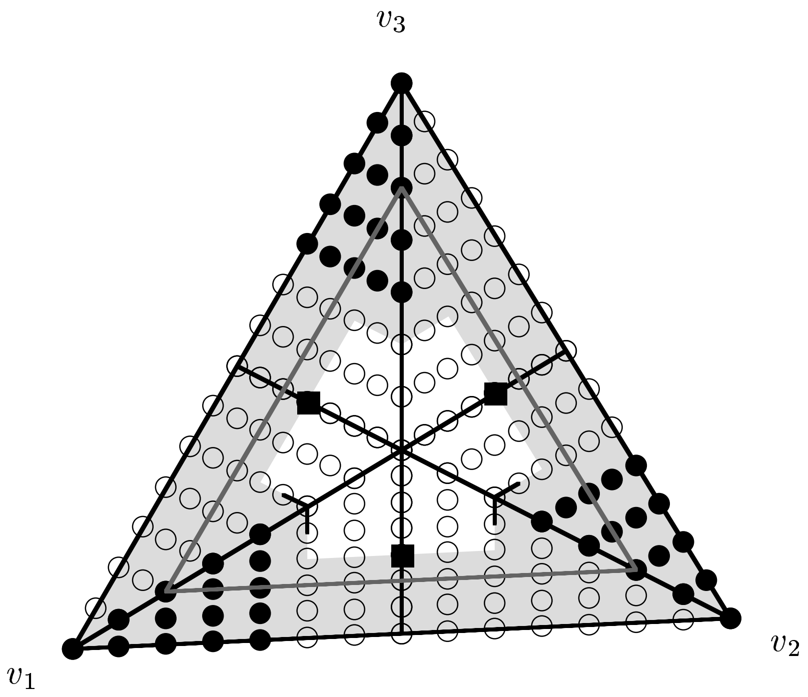

Fig. 3. The macro-element $\mathcal{S}_3(T_{PS})$.

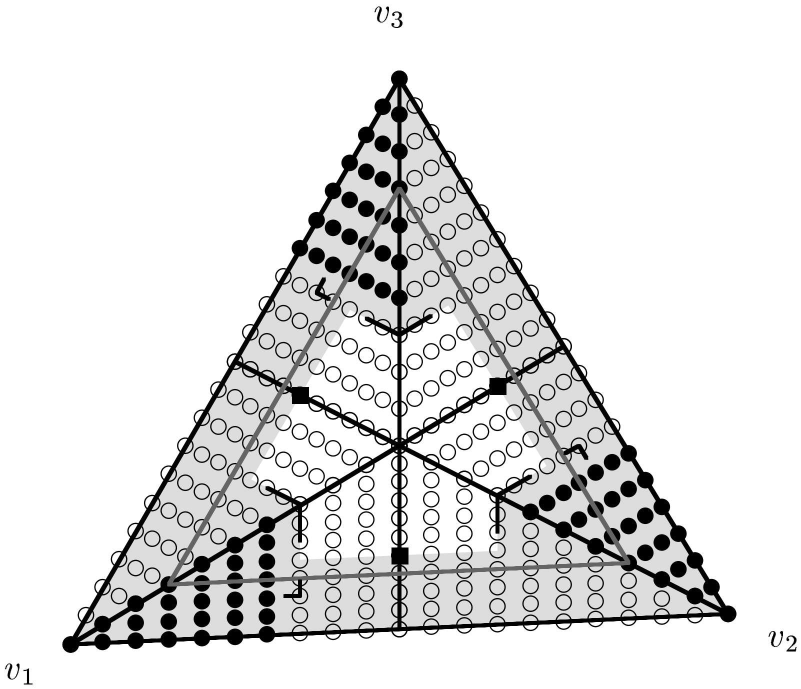

Fig. 4. The macro-element $\mathcal{S}_4(T_{PS})$.

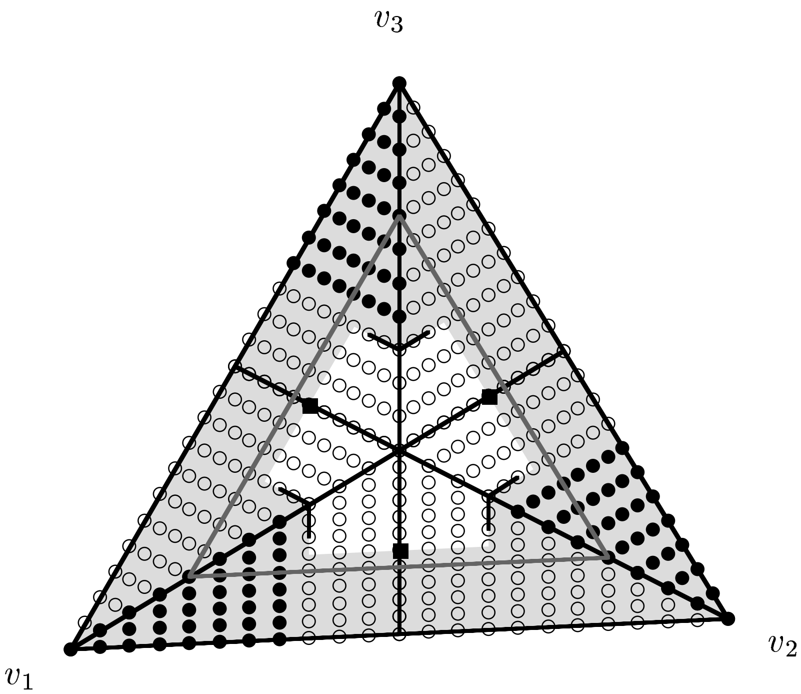

Fig. 5. The macro-element $\mathcal{S}_5(T_{PS})$.

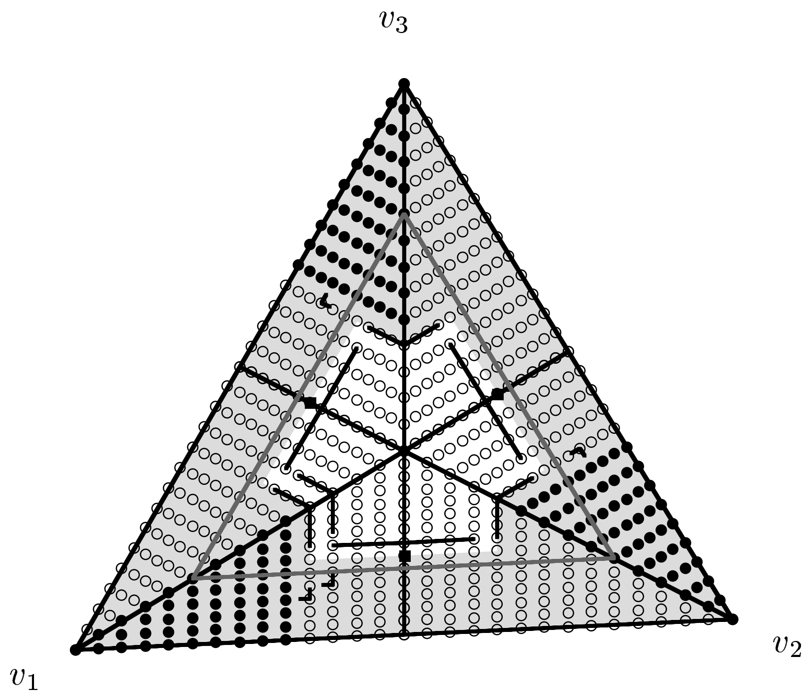

Fig. 6. The macro-element $\mathcal{S}_6(T_{PS})$.

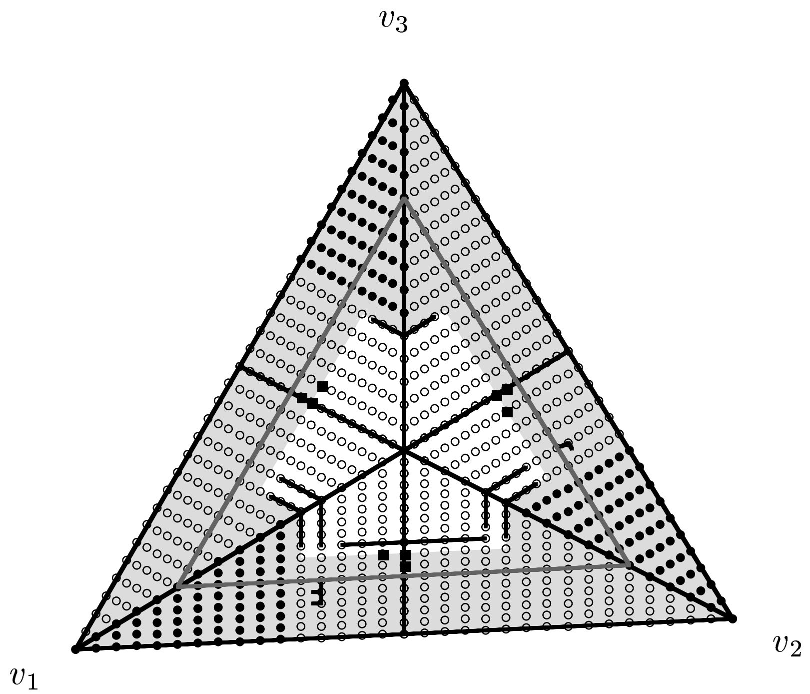

Fig. 7. The macro-element $\mathcal{S}_7(T_{PS})$.

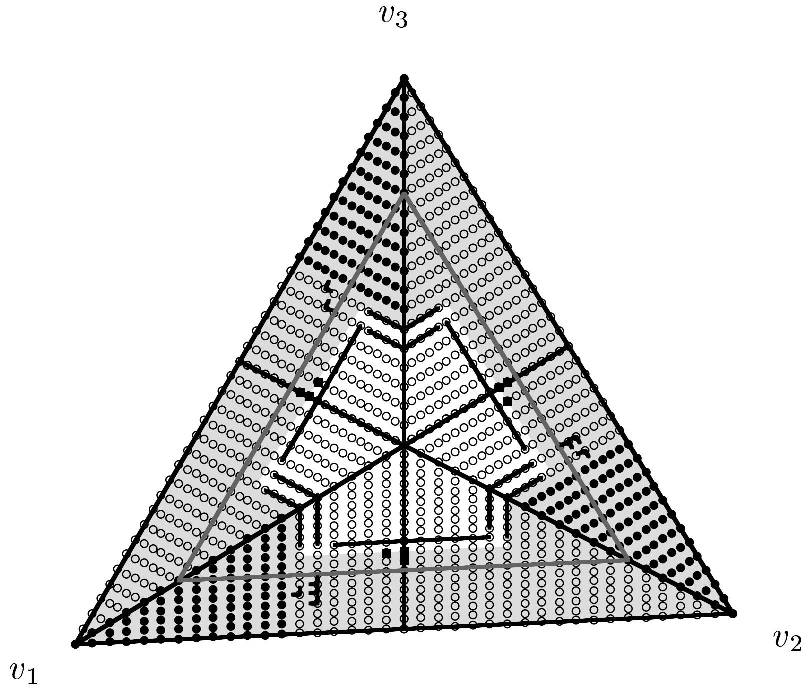

Fig. 8. The macro-element $\mathcal{S}_8(T_{PS})$.

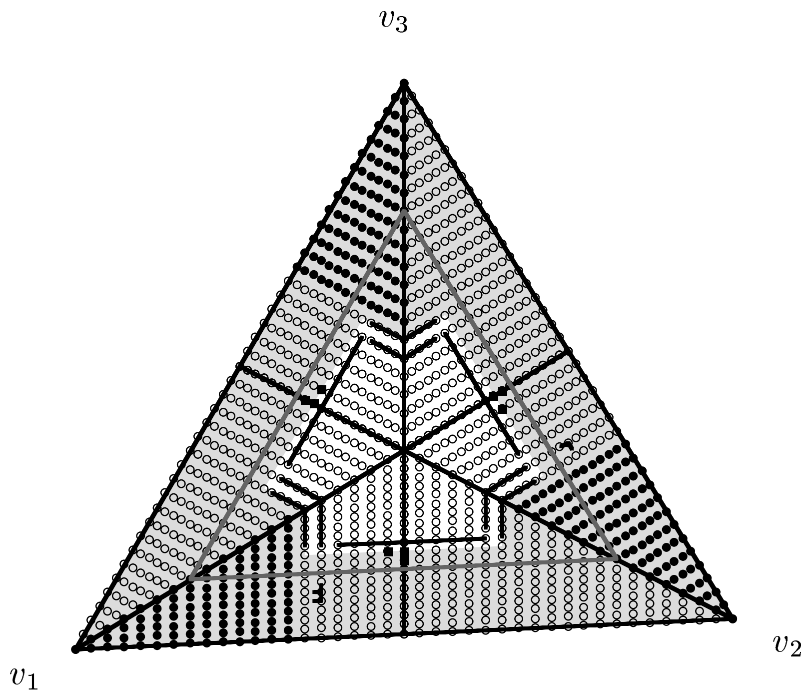

Fig. 9. The macro-element $\mathcal{S}_9(T_{PS})$.

Theorem 7.1. *Given $8k \le r \le 8k+7$, let ρ, μ, d be as in Table 1. Then*

$$\dim \mathcal{S}_r(\triangle_{PS}) = \binom{3\rho+2}{2} V + \binom{r+\mu-d+1}{2} E, \tag{7.2}$$

where V and E are the number of vertices and edges of $\triangle$, respectively.

Proof: Let $\mathcal{M}_r$ be the following set of domain points:

1) for each vertex v of $\triangle$, choose a triangle T of $\triangle_{PS}$ attached to v and include $D_\rho^T(v)$,
2) for each edge $e = \langle v_1, v_2 \rangle$ of $\triangle$, let $T = \langle v, v_1, v_2 \rangle$ be a triangle of $\triangle_{PS}$ containing the edge e. Then include the points $\{\xi^T_{j,j+\mu-d-1,2d+1-\mu-2j}, \ldots, \xi^T_{j,0,d-j}\}$ for $j = d-\mu+1, \ldots, r$.

The cardinality of $\mathcal{M}_r$ is precisely the number in (7.2). Now setting the coefficients c_ξ of s for $\xi \in \mathcal{M}_r$, we can use the smoothness conditions to uniquely determine all remaining coefficients in the disks $D_\rho(v_i)$. Then the remaining coefficients in each macro-triangle can be uniquely computed as in the proof of Theorem 4.2. This shows that $\mathcal{M}_r$ is a MDS, and the result follows. □

We claim that the minimal determining set $\mathcal{M}_r$ described in the proof of Theorem 7.1 is stable. To see this, we note that as shown in Lemma 2.2 of [13], the smallest angle in the refined triangulation $\triangle_{PS}$ is at least $\theta_\triangle \sin(\theta_\triangle)/4$, where $\theta_\triangle$ is the smallest angle in $\triangle$.

We are now ready to solve the Hermite interpolation problem.

Theorem 7.2. *Given $r \ge 0$, let d, ρ, μ be as in Table 1. For any function f which is sufficiently smooth so that the needed derivatives exist, there is a unique spline $s \in \mathcal{S}_r(\triangle_{PS})$ such that*

$$D_x^\nu D_y^\mu s(x_i, y_i) = D_x^\nu D_y^\mu f(x_i, y_i), \qquad 0 \le \nu + \mu \le \rho, \qquad i = 1, \ldots, n,$$

and

$$D_e^{d-\mu+j} s(\eta^j_{e,i}) = D_e^{d-\mu+j} f(\eta^j_{e,i}), \qquad 1 \le i \le j, \quad 1 \le j \le r-d+\mu,$$

for all edges e of $\triangle$, where $\eta^j_{e,i}$ are the points associated with e as in (6.1).

Proof: For each triangle T of $\triangle$, the interpolant s can be constructed locally since the given data uniquely determines the nodal data listed in Theorem 6.1. By construction of the macro-elements, the resulting spline s is globally in C^r, and thus lies in the spline space $\mathcal{S}_r(\triangle_{PS})$. □

The Hermite interpolant of Theorem 7.2 is exact for polynomials of degree d. Coupling this with the stability of the construction and using the Bramble-Hilbert lemma as in [12], it is easy to establish the following error bound which shows that for sufficiently smooth f, the Hermite interpolant provides optimal order approximation.

Theorem 7.3. *Suppose f lies in the Sobolev space $W_\infty^{k+1}(\Omega)$ for some $\rho \le k \le d$, and let s be the interpolating spline of Theorem 7.2. Then*

$$\|D_x^\alpha D_y^\beta (f - s)\|_\infty \le K|\triangle|^{k+1-\alpha-\beta}|f|_{k+1,\infty} \tag{7.3}$$

for $0 \le \alpha + \beta \le k$, where $|\triangle|$ is the mesh size of $\triangle$ (i.e., the diameter of the largest triangle), and $|f|_{k+1,p}$ is the usual Sobolev semi-norm. If Ω is convex, then the constant K depends only on r and on the smallest angle $\theta_\triangle$ in $\triangle$. If Ω is nonconvex, it also depends on the Lipschitz constant $L_{\partial\Omega}$ associated with the boundary of Ω.

Although we did not need a basis to solve the Hermite interpolation problem, for other applications it is useful to observe that the space $\mathcal{S}_r(\triangle_{PS})$ has a convenient stable local basis. For each ξ in the MDS $\mathcal{M}_r$ defined in the proof of Theorem 7.1, let B_ξ be the unique spline in $\mathcal{S}_r(\triangle_{PS})$ such that

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \eta \in \mathcal{M}_r, \tag{7.4}$$

where λ_η is the linear functional which picks off the B-coefficient corresponding to the domain point η. In view of (7.4), the splines $\{B_\xi\}_{\xi\in\mathcal{M}_r}$ are linearly independent, and thus form a basis for $\mathcal{S}_r(\triangle_{PS})$. It is easy to see that the B_ξ have local support. In particular,

1) If ξ is a point as in item 1) of Theorem 7.1, then $\mathrm{supp}(B_\xi)$ is contained in the union of all triangles of $\triangle$ sharing the vertex v.
2) If ξ is a point as in item 2) of Theorem 7.1, then $\mathrm{supp}(B_\xi)$ is contained in $T \cup \widetilde{T}$, where T and $\widetilde{T}$ are the triangles of $\triangle$ sharing the edge e. (If e is a boundary edge of $\triangle$, then there is only one such triangle, and it is the support set).

§8. Remarks

Remark 8.1. Macro-elements can be constructed without splitting triangles, see [23–26]. As observed in [22], they belong to the classical superspline space $\mathcal{S}_{4r+1}^{r,2r}(T)$, see also [15,16].

Remark 8.2. The java code of the first author for examining determining sets for superspline spaces was the key tool in discovering the macro-elements described in this paper. The code is described in [1], and can be used or downloaded from `http://www.math.utah.edu/~alfeld`. That web site also contains code that can be used to generate colored versions of our figures for all values of r, along with a detailed documentation of the MDS code.

Remark 8.3. As shown in [13], it is not possible to construct C^r macro-elements on the PS-split using splines of lower degree than those considered here, and it is not possible to enforce lower supersmoothness at the vertices of T. At first glance it might appear the degrees of freedom in item 2) of Theorem 4.2 could also be removed. This can in fact be done if one is interested only in an element on a single triangle. But it does not work if we want an element which can be applied locally in a larger triangulation. The reason is that enforcing any additional smoothness will lead to a coefficient in 2) being determined by data at all three vertices. If this is done on two neighboring macro triangles, we will in general get coefficients which do not satisfy the C^r smoothness across the common edge.

Remark 8.4. It is of course possible to construct smooth macro-elements on other triangle splits such as the Clough-Tocher split. For macro-elements on these splits, see [2,8,12]. The degrees of the splines used here are lower than for the Clough-Tocher splits, but the cost is a more complicated split. Splines of still lower degree could be obtained by working with even more complicated triangle splits.

Remark 8.5. It is also possible to create macro-elements with even fewer degrees of freedom by the process of condensation. This amounts to further restricting the spline space (usually by forcing certain cross-derivatives along edges of the triangle T to be of lower degree than they naturally are). We do not recommend this strategy since it produces elements which no longer reproduce the full polynomial space, and thus have reduced approximation power.

Remark 8.6. The construction described here is not unique in the sense that there are other choices of the extra smoothness conditions which also lead to macro-elements with the same number of degrees of freedom.

Remark 8.7. In contrast to the Clough-Tocher case [2], it is not possible to perform Powell-Sabin refinement on general triangulations by splitting all triangles $T := \langle u, v, w\rangle$ about a split point $v_T := ru + sv + tw$ with *fixed* barycentric coordinates r, s, t. In Sect. 7 we have followed the usual approach in which the points v_T are chosen to the be the incenters of the triangles. This is known to guarantee that refinement is possible and also insures the stability.

Remark 8.8. Theorem 7.3 describes the approximation power of the the spaces $\mathcal{S}_r(\triangle_{PS})$ measured in the uniform norm. Analogous results hold for the p-norms, and can be proved using the quasi-interpolation operators Q_k defined by

$$Q_k f := \sum_{\xi\in\mathcal{M}_r} \lambda_{\xi,k} f\, B_\xi,$$

where B_ξ are the dual basis splines of Sect. 6 and $\lambda_{\xi,k}$ are the linear functionals defined in Sect. 10 of [11].

Remark 8.9. For special results on Powell-Sabin elements associated with uniform three-direction (type-I) meshes, see [6–7].

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