Bivariate Spline Spaces and Minimal Determining Sets

by

Peter Alfeld,
Department of Mathematics,
University of Utah,
Salt Lake City, Utah 84112, U.S.A.
alfeld@math.utah.edu,
http://www.math.utah.edu/~alfeld/

Abstract

The paper describes bivariate spline spaces and minimal determining sets, shows applications of these concepts, presents two new triangular finite elements, describes public domain software for the analysis of spline spaces, and concludes with a concise history of these subjects. The software described in this paper is available on

http://www.math.utah.edu/~alfeld/MDS

That page also contains links to colored versions of the Figures in this paper.

1. Introduction

A large role in Larry Schumaker's and my own work, jointly and separately, has been played by the concept of a *minimal determining set*. In this paper, we define that concept, illustrate its utility, and describe a Java based piece of software that is available on the world wide web and can be used by anybody for teaching or research purposes.

To understand this paper familiarity with the Bernstein-Bézier form of a bivariate polynomial is essential. The concepts are standard and described and defined in many of Schumaker's papers, e.g., [24], and a forthcoming monograph by Lai and Schumaker, [19].

To help make this paper self contained the remainder of this section contains a brief statement of the relevant definitions.

Definition 1. *Let T denote a non-degenerate triangle with vertices V_1, V_2, V_3 and $x \in \mathbb{R}^2$ a point. The barycentric coordinates b_1, b_2, b_3 of x (with respect to T) are defined by*

$$x = \sum_{i=1}^{3} b_i V_i, \qquad \sum_{i=1}^{3} b_i = 1. \tag{1}$$

The Bernstein-Bézier form of a polynomial p of degree d is given by

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k. \tag{2}$$

The c_{ijk} are the Bézier ordinates of p, and the points

$$P_{ijk} = \frac{iV_1 + jV_2 + kV_3}{d} \tag{3}$$

its domain points. The points

$$(P_{ijk}, c_{ijk}), \qquad i+j+k=d \tag{4}$$

are the control points of p.

Definition 2. *Let*

$$\mathcal{V} = \left\{V_i \in I\!R^2,\ i = 1, 2, \ldots, V\right\} \tag{5}$$

be a given set of vertices. A set of triangles

$$\triangle = \{T_i, i = 1, \ldots, N\} \tag{6}$$

is a triangulation of $\mathcal{V}$ provided

1. *The interiors of all triangles in $\triangle$ are pairwise disjoint.*
2. *Each edge of a triangle in $\triangle$ is either a common edge of two triangles in $\triangle$ or else it is on the boundary of*

$$\Omega = \bigcup_{T \in \triangle} T. \tag{7}$$

3. *Ω is homeomorphic to the square.*

With every triangulation $\triangle$ we associate the following parameters:

$$\begin{aligned}
V_B &= \text{the number of boundary vertices of } \triangle \\
V_I &= \text{the number of interior vertices of } \triangle \\
V &= V_B + V_I \\
E_B &= V_B = \text{the number of boundary edges of } \triangle \\
E_I &= V_B + 3V_I - 3 = \text{the number of interior edges of } \triangle \\
E &= E_B + E_I \\
N &= V_B + 2V_I - 2 = \text{the number of triangles in } \triangle
\end{aligned} \tag{8}$$

The equalities in (8) can be easily verified by induction in N, see also [13].

The following definition introduces the central object of this paper:

Definition 3. *Given a set $\mathcal{V}$ of vertices and a triangulation $\triangle$ of $\mathcal{V}$,*

$$S_d^r(\triangle) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d\ \forall T \in \triangle\} \tag{9}$$

where $\mathcal{P}_d$ denotes the $\binom{d+2}{2}$ dimensional space of bivariate polynomials of degree d. The elements of $S_d^r(\triangle)$ are called (bivariate) splines.

Thus $S_d^r(\triangle)$ is the linear space of all functions that on each triangle can be represented as a bivariate polynomial of degree d and that are r times differentiable everywhere on the domain Ω. This is the natural generalization of the concept of smooth piecewise polynomial functions used ubiquitously in numerical analysis for the solution of problems involving univariate functions (of one independent variable).

When statements apply to any triangulation or there is otherwise no confusion we omit the triangulation and simply use the notation S_d^r.

Two polynomials defined on neighboring triangles will join *continuously* iff the two sets of control points on the common edge coincide. We will thus always identify control points on shared edges. With that identification the control points uniquely define and are defined by a given spline in S_d^0. The dimension of that space is

$$\delta := \dim S_d^0 = V + (d-1)E + \binom{d-1}{2}N. \tag{10}$$

Of crucial importance is the dimension of S_d^r. Any function in S_d^r can be thought of as a function in S_d^0 that satisfies additional homogeneous *smoothness conditions*.

Let T denote a triangle with vertices V_1, V_2, V_3, as before, and let $\tilde{T}$ denote a triangle with vertices V_2, V_3, and V_4. We write a polynomial p on T in its Bernstein-Bézier form (2), and a polynomial $\tilde{p}$ on $\tilde{T}$ similarly, with vertex V_4 replacing V_1 and the barycentric coordinates and Bézier ordinates distinguished by tildes. Let γ_1, γ_2, and γ_3 denote the barycentric coordinates of V_4 with respect to T. Then, in order for p and $\tilde{p}$ to join r times differentiably across the common edge V_2V_3 we must have that

$$\tilde{c}_{\rho jk} - \sum_{\mu+\nu+\kappa=\rho} \frac{\rho!}{\mu!\nu!\kappa!} c_{\mu,j+\nu,k+\kappa}\gamma_1^\mu \gamma_2^\nu \gamma_3^\kappa = 0, \tag{11}$$

for all $\rho \le r$ and $j + k + \rho = d$.

The smoothness equations (11) for all pairs of triangles sharing an interior edge can be collected into a homogeneous linear system

$$Ac = 0 \tag{12}$$

where A is an $s \times \delta$ matrix with

$$s = \frac{r(2d - r + 1)}{2} E_I. \tag{13}$$

The entries of A are rational functions of the coordinates of the vertices of $\triangle$.

Thus the dimension of S_d^r is given by

$$\dim S_d^r = \delta - \operatorname{rank} A. \tag{14}$$

Let $\mathcal{M}$ denote the set of domain points of a spline in S_d^0, and c_P the Bézier ordinate corresponding to the point $P \in \mathcal{M}$.

Definition 4. *A subset $\mathcal{D}$ of $\mathcal{M}$ is a determining set for S_d^r if*

$$c_P = 0 \ \forall P \in \mathcal{D} \quad \Longrightarrow \quad c_P = 0 \ \forall P \in \mathcal{M}. \tag{15}$$

$\mathcal{D}$ is a minimal determining set if there is no smaller determining set.

The significance of minimal determining sets is captured in the following properties which follow from elementary linear algebra and are stated here without proof:

1. The dimension of S_d^r equals the number of points in a minimal determining set $\mathcal{D}$.
2. Given a minimal determining set $\mathcal{D}$, a basis of S_d^r is given by the set

$$\mathcal{B} = \{s_P \in S_d^r : \quad P \in \mathcal{D}, \quad c_P = 1, \quad c_Q = 0 \quad \forall P \neq Q \in \mathcal{D}\}. \tag{16}$$

The splines in $\mathcal{B}$ are often called *cardinal splines.*

We conclude this section with some remarks:

Remark 5. *The crucial difference between univariate and bivariate (or, more generally, multivariate) splines is that the dimension of S_d^r may change as the location of the vertices is perturbed by an arbitrarily small amount. By contrast, in the univariate analog (of smooth piecewise polynomial functions defined on the partition of an interval) the dimension of spline space depends only on the number of subintervals. The dependence of the dimension of bivariate spline spaces on the geometry of the triangulation was first observed in [20].*

Remark 6. *Every spline space $S_d^r(\triangle)$ possesses a generic dimension* G *which is such that if* $\dim S_d^r(\triangle) \neq \mathrm{G}$ *then*

- $\dim S_d^r(\triangle) > \mathrm{G}$, *and*
- *there exists a triangulation $\triangle^*$ which can be obtained from $\triangle$ by an arbitrarily small perturbation of the location of the vertices of $\triangle$ such that* $\dim S_d^r(\triangle^*) = \mathrm{G}$.

Remark 7. *Similarly as the dimension, the solvability of an interpolation problem may depend on the location of the vertices.*

Remark 8. *Explicit minimal determining sets (and hence bases of the underlying spline space) have been constructed for the following cases:*

1. *Cells, i.e., stars of vertices [23].*
2. *$S_d^r(\triangle)$ for general triangulations $\triangle$ in the case $d \geq 3r+2$, [16], [18].*
3. *$S_d^r(\triangle)$ for almost all triangulation in the case $d = 3r+1$, [6].*
4. *$S_4^1(\triangle)$ for all triangulations $\triangle$, [5].*

By contrast, if $d \leq 3r$ very little is known. The following two conjectures provide extremely challenging research problems. Their confirmation (or refutation) has withstood determined efforts by a large number of people (including Larry Schumaker and this author) since Gilbert Strang first recognized the problems in 1973, [25].

Before stating the conjectures we need to define the concept of a *singular vertex.* A singular vertex is an interior vertex with four edges attached which form two parallel pairs, [20]. Let σ denote the number of singular vertices of a triangulation $\triangle$.

Conjecture 9.

$$\dim S_3^1(\triangle) = 3V_B + 2V_1 + 1 + \sigma. \tag{17}$$

Remark 10. *It is known, [21], that*

$$\dim S_3^1(\triangle) \geq 3V_B + 2V_1 + 1 + \sigma. \tag{18}$$

Conjecture 11. *Given a triangulation $\triangle$ and numbers F_i, $i = 1, 2, \ldots, V$, there exists a function $s \in S_3^1$ such that*

$$s(V_i) = F_i, \qquad i = 1, 2, \ldots, V. \tag{19}$$

(In other words one can always interpolate function values at the vertices of a triangulation with S_3^1.)

2. Examples of Minimal Determining Sets

We illustrate minimal determining sets with some examples.

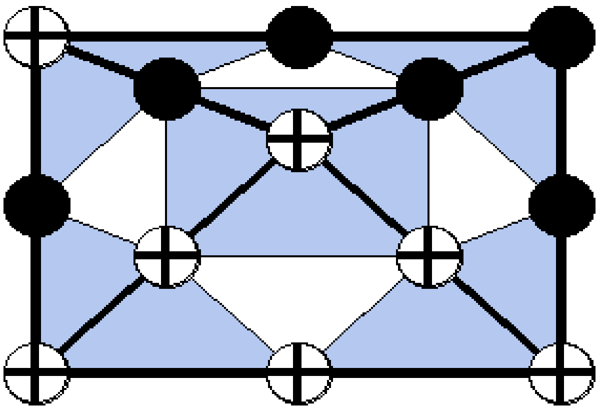

Figure 1. A generic 4-star.

Example 12, a singular vertex. *This example is first mentioned in [20]. Consider the space S_2^1 on the star of an interior vertex of degree 4. Figure 1 shows a minimal determining set of 7 points for this space, in the case that the number of slopes of edges emanating from the interior vertex is at least 3. The points in the minimal determining set are marked with plus signs contained in circles. The remaining points are marked with solid circles. The values of their Bézier ordinates are implied by the Bézier ordinates corresponding to points in the minimal determining set. By comparison, if the interior vertex is singular, as indicated in Figure 2, then the dimension of S_2^1 is 8, i.e., it is one larger than the generic dimension. This phenomenon can be explained geometrically using the fact that each quadrilaterals formed by two neighboring triangles has degenerated into a triangle.*

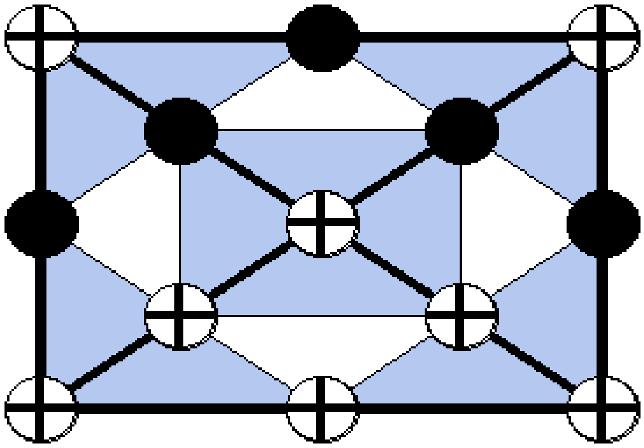

Figure 2. A singular Vertex.

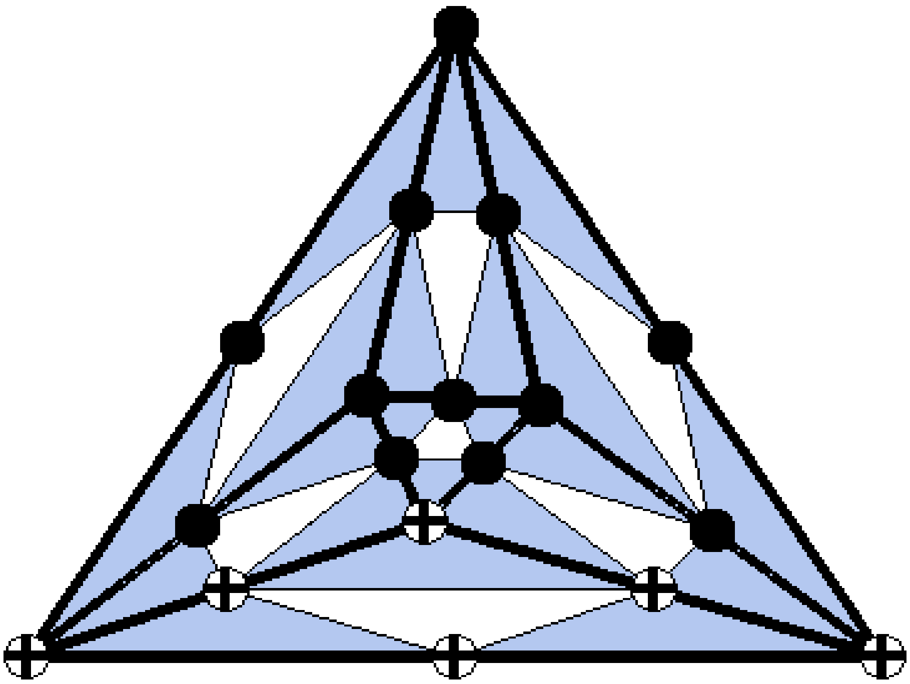

Figure 3. The generic Morgan-Scott Split.

Example 13, the Morgan-Scott split. *This example is also taken from [20] where the case of S_2^1 is considered. Consider the generic partition of a triangle into seven subtriangles, as indicated in Figure 3. The dimension of S_2^1 on this split is 6. However, if the split is*

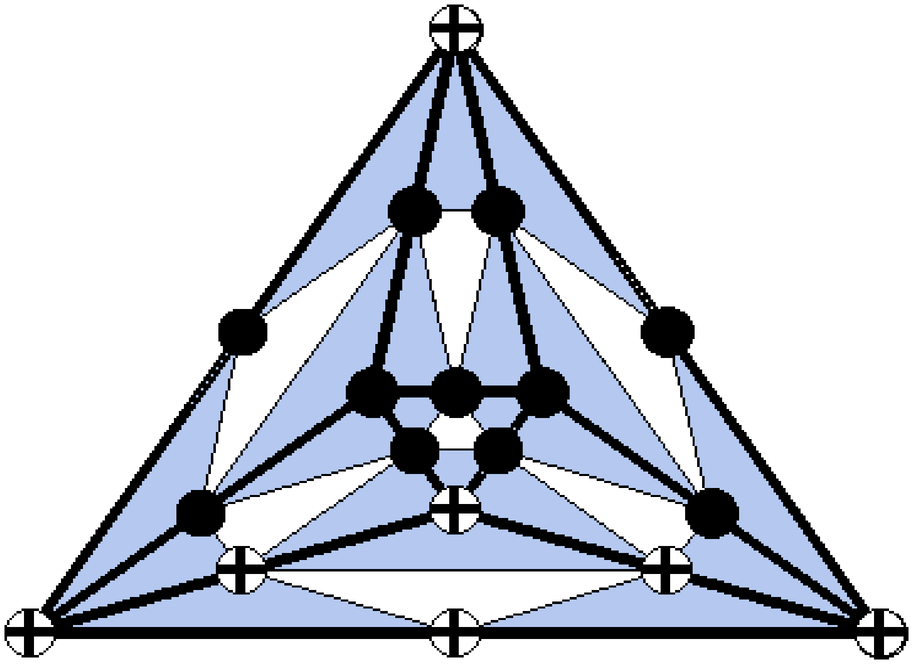

Figure 4. The symmetric Morgan-Scott Split.

symmetric, as in Figure 4, the dimension increases to 7. Diener, [12], later showed that in general this happens when $d = 2r$. Figures 3 and 4 illustrate minimal determining sets. To define symmetry in this context, consider the three lines from the vertices of the outer triangle to the opposite vertex of the inner triangle. The split is symmetric if and only if these three lines all intersect in the same point. The degeneracy in this example cannot be explained in terms of a single vertex star. For examples of more complex triangulations that lead to non-generic dimensions see [2].

Example 14, Confinable Vertices. *In [5] we needed to investigate the question of under which circumstances one can interpolate with S_2^1 at the boundary vertices of a vertex star. When this is possible the interior vertex is said to be* confinable*. We found just one example of a non-confinable vertex, a regular hexagon. Figure 5 demonstrates that specifying the function value at five boundary vertices implies the function value at the sixth boundary vertex. The dimension of S_2^1 on this triangulation is 9, but no minimal determining set exists that includes all six boundary vertices. The non-confinability is more subtle than one would first expect. It is not simply caused by the the interior edges forming three parallel pairs. Rather it is also necessary that all six angles are equal, or else the triangulation is obtained by applying the same affine transformation to all vertices of a hexagon with six equal angles.*

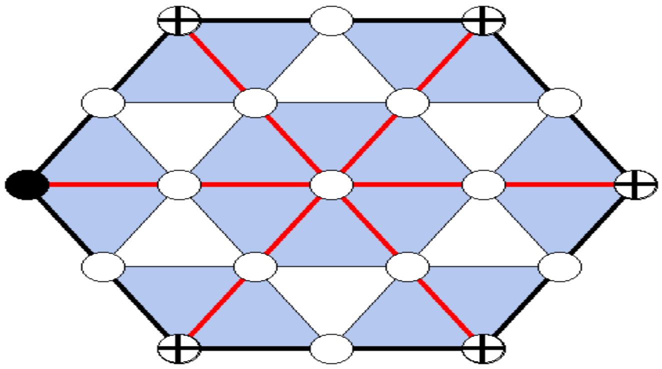

Figure 5. A non-confinable Vertex.

Example 15, a C^2 finite element. *Figure 6 illustrates a C^2 finite element defined on the double Clough-Tocher split. This scheme is superior to the one proposed in [1]. Details will be discussed in a forthcoming paper, [8]. The element was first discovered using the software described below. Figure 6 shows how the macrotriangle is split into 9 microtriangles. On this triangulation consider that subspace of S_5^2 of all functions that are three times differentiable at the centroid of the macrotriangle. Points marked with plus signs in circles denote Bézier ordinates that are determined by interpolation to function, gradient, and Hessian values at the vertices of the macrotriangles. Points marked with hollow circles indicate Bézier ordinates that are chosen so as to ensure C^2 transitions between neighboring macrotriangles. Bézier ordinates at points marked with hollow boxes are chosen to interpolate to function values at the centroid and the subcentroids. The three Bézier ordinates tat the points marked with filled boxes are chosen so as to interpolate to certain third order derivatives at the vertices of the macrotriangle. Bézier ordinates at the remaining points, marked with small dots, are implied by the smoothness conditions. The resulting scheme is piecewise quintic, globally C^2, and it reproduces quintic functions exactly.*

Example 16, a C^3 element. *Figure 7 shows a similar C^3 element of polynomial degree 7, which will also be discussed in [8].*

3. Constructing a Minimal Determining Set

Constructing a minimal determining set is tantamount to constructing a basis and central to the analysis of spline spaces. Java [17] based software is available on

`http://www.math.utah.edu/~alfeld/MDS`

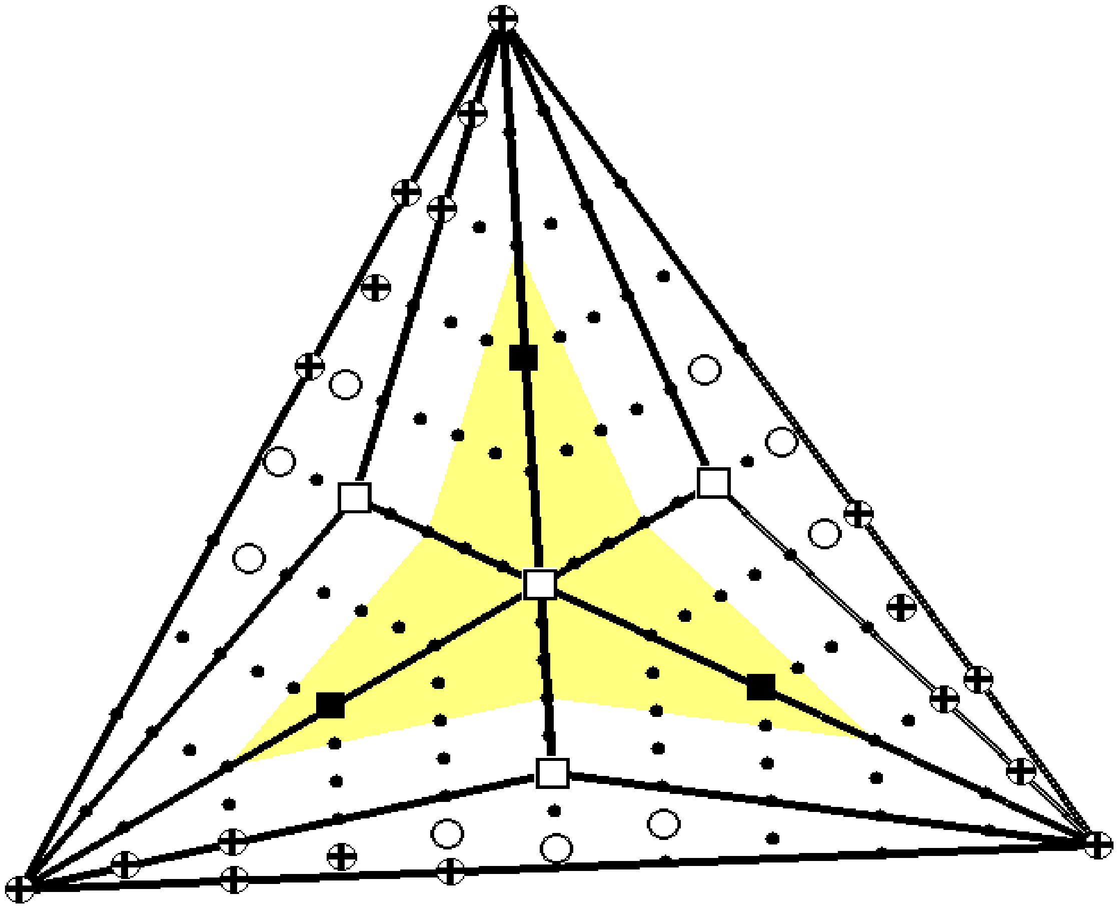

Figure 6. A double Clough-Tocher C^2 Finite Element.

that you can use to construct a minimal determining set interactively by adding and removing points from a growing set that eventually will turn into a minimal determining set. You simply click on appropriate domain points and the software will mark those points whose Bézier coefficients are implied by the points you have already chosen. (You can also generate drawings, analyze spline spaces on a number of built-in triangulations, or design and edit your own triangulations.)

This section describes some of the mathematical basis of the software. To appreciate the section it is best to link (with a Java enabled browser) to the above mentioned URL and use the software on a few examples. Simply click on the applet and then click on the domain points of the spline space that will appear in a window on your screen. The web page has numerous links to supporting documentation, including a detailed User's Guide. For serious work it is preferable to download the software rather than use it through a browser.

Constructing a minimal determining set requires an analysis of the linear system (12).

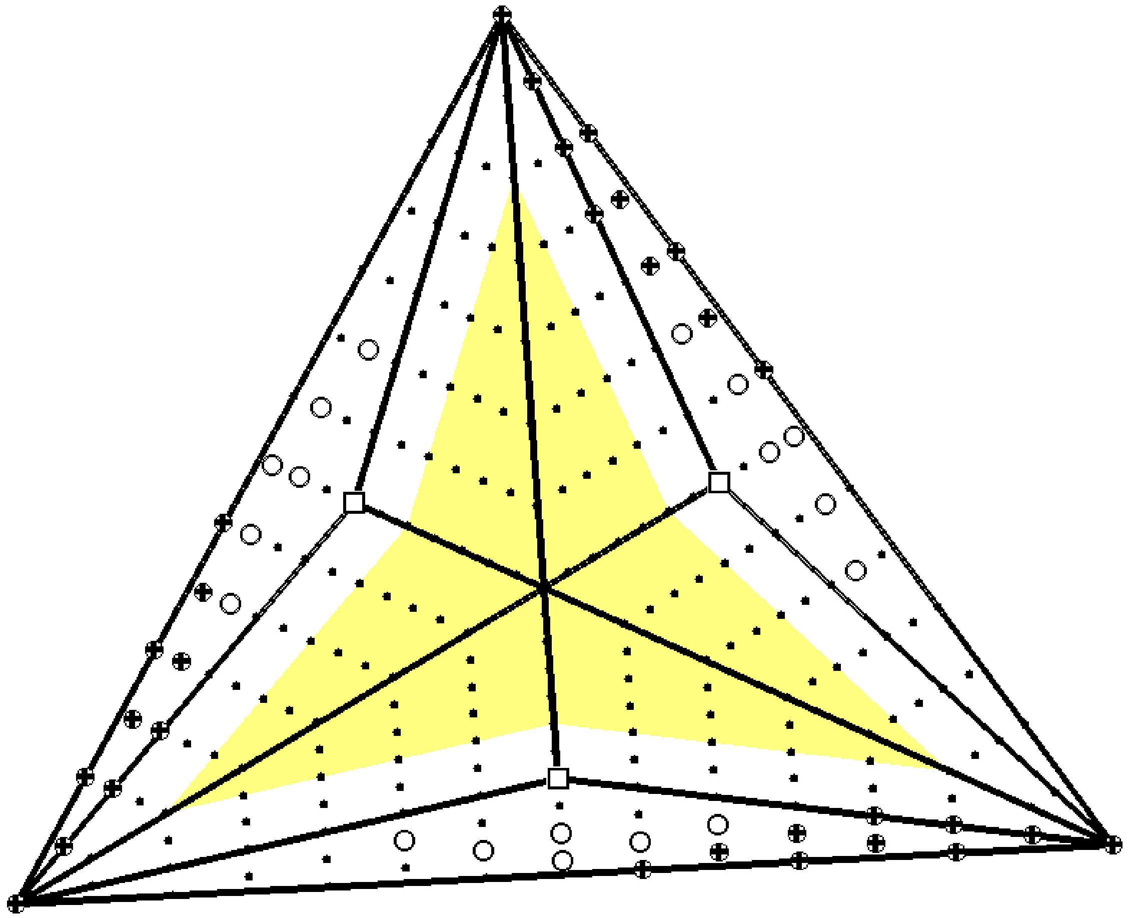

Figure 7. A double Clough-Tocher C^3 Finite Element.

Crucial attributes are the rank of the system and the dependence of certain sets of smoothness conditions. Analyzing such dependencies requires Gaussian Elimination. For our application, this is virtually impossible in floating point arithmetic since in spline analysis we are particularly interested in geometric degeneracies, corresponding to increased rank deficiencies of A, that are easily obscured by inexact computations.

However, the special nature of (12) allows an alternative approach. The vertices of the triangulation are pixels, and the coordinates of the pixels are *integers*. Thus the entries of A in (12) in its natural formation are *rational.* The system is made integer by multiplying each row with the least common denominator in that row.

Instead of the integers themselves, which could grow quite large in the process of manipulating the system, the code keeps the residuals of the entries with respect to one or more prime numbers. Usually there is just one such number, specifically,

$$p = 2^{31} - 1 = 2,147,483,647 \tag{20}$$

(which is the largest prime number that can be expressed as a long integer in Java).

The residual of n modulo p is the remainder of n/p. If a number (such as the determinant of a square submatrix) is zero, the residual will be zero. On the other hand, the residual is also zero if n is a multiple of p. Thus the rank of A will never be overestimated, but it is possible that it is underestimated. Because of the size of p the likelihood of an accidentally zero residual is slim. For a single number, assuming all residuals are equally likely, it is $1/p$. Moreover, one can show via Hadamard's inequality and the Chinese Remainder Theorem, [9], that by doing the analysis modulo sufficiently many primes one can always be sure that no entry of the system is mistaken for zero.

For a more detailed discussion of residual arithmetic see the appropriate link on the MDS home page. For simplicity, in the remainder of this section we describe the analysis as if it were carried out in ordinary integer arithmetic.

The MDS code builds a minimal determining set M adding one point at a time. For both research and teaching purposes it is useful not just to add and but also to subtract points from the growing set. At every change the code updates the display and marks points whose Bézier ordinates are implied by the already chosen point in the minimal determining set, and unmarks any points that are no longer implied.

The procedure is very similar to the Simplex Algorithm for solving linear programming problems, [15].

As before, consider the system (12) where s is the number of smoothness conditions and δ is the total number of domain points. (As a practical matter we omit points deep in the interior of triangles that do not enter any smoothness conditions. They give rise to zero columns in A and must be included in any minimal determining set.) As described above, we assume the entries of A are *integer.*

A division free version of Gaussian Elimination (ignoring for the moment the need to pivot) is given by this algorithm:

$$\begin{aligned}
&\text{For } k = 1, \dots, s-1 \\
&\quad \text{For } i = 1, \dots, \delta \\
&\qquad \mathrm{m} = a_{ik} \\
&\qquad a_{ik} = 0 \\
&\qquad \text{For } j = k, \dots, \delta \\
&\qquad\quad a_{ij} = a_{kk} a_{ij} - \mathrm{m} a_{kj}
\end{aligned} \tag{21}$$

Clearly this gives $a_{ik} = 0$. Since the process does not involve any divisions it can always be carried out. However, if $a_{kk} = 0$ then the i-th row will be replaced with a multiple of the k-th row and we lose one of the equations. Thus we still need to require that

$$a_{kk} \neq 0. \tag{22}$$

But it is no longer necessary to pivot for stability, i.e., the first non-zero pivot may be used.

We begin by using this variant of Gaussian Elimination with row operations and column interchanges to transform A into a new matrix

$$A \longrightarrow \begin{bmatrix} D & A_M \\ 0 & 0 \end{bmatrix} \tag{23}$$

where D is an $m \times m$ diagonal matrix and A_M is an $m \times (\delta - m)$ matrix with integer entries. We denote the (i,i)-entry of D by d_i, i.e.,

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_m \end{bmatrix} \tag{24}$$

The dimension of S_d^r is

$$\dim S_d^r = \delta - m. \tag{25}$$

The points corresponding to columns of A_M form a minimal determining set.

We now think of I and M as each being composed of two sets:

$$M = M_1 \cup M_2 \tag{26}$$

where M_1 is the set of points we have already picked to be in the minimal determining set, and M_2 is a set of points whose status is as yet unknown. Initially M_1 is empty and $M_2 = M$.

Similarly, we write

$$I = I_1 \cup I_2 \tag{27}$$

where I_1 is the set of points that we know to be determined by our growing minimal determining set M_1, and I_2 is a set of points whose status we do not yet know. Initially, I_1 is empty, and $I_2 = I$.

We think of the rows of A_M as labeled by the indices in I and the columns by the points in M. For points $p \in I$ and $q \in M$ the notation a_{pq} means the entry in M in the location (i_p, j_q) where i_p and j_q are the physical row and column indices associated with p and q.

Criterion 17. *A point $p \in I$ is in I_1 iff the row of M_2 corresponding to p is zero.*

The "if" direction is obvious. The "only if" direction follows from the observation below that if that row in M_2 is non-zero then we can add p to our growing minimal determining set.

We consider the following cases:

Case 18: We pick $p \in M_2$ to be a new point in the minimal determining set.

We transfer p from M_2 to M_1, and check if this implies any additions to I_1 (by seeing if we have any new zero rows in the reduced M_2).

Case 19: We pick $p \in I_2$ to be a new point in the minimal determining set.

In that case we exchange p for a point $q \in M_2$ and change the matrix M appropriately. We denote new entries of M by bars.

First we find a point $q \in M_2$ such that

$$a_{pq} \neq 0. \tag{28}$$

This is always possible because otherwise p would be in I_1, i.e., it would be determined by the points in the set M_1 that we have already picked.

In principle, any point satisfying (28) will do. This is distinct from floating point arithmetic where for the sake of numerical stability one maximizes the absolute value of a_{pq}.

However, there is a subtlety in the case of residual arithmetic. We need a point q such that the residual of $a_{pq} \neq 0$ for *all* primes being used. The current code assumes that such a point can always be found. If not, we could drop the prime(s) that do give rise to a zero residual, and perhaps replace them by others, but that option is not yet implemented.

The equation in the row corresponding to p reads:

$$d_p c_p + \sum_{\substack{t \in M \\ t \neq q}} a_{pt} c_t + a_{pq} c_q = 0. \tag{29}$$

In this row we make no change, except that since we will exchange p and q, d_p and a_{pq} exchange their roles (and maintain their numerical value).

The equation corresponding to $w \neq p$ (or q after the exchange of p and q) reads similarly:

$$d_w c_w + \sum_{\substack{t \in M \\ t \neq q}} a_{wt} c_t + a_{wq} c_q = 0. \tag{30}$$

We want to eliminate the entries in the column corresponding to q. To this end we replace the equation (30) with

$$q_{wq}(29) - a_{pq}(30) \tag{31}$$

which gives:

$$a_{wq} \left(d_p c_p + \sum_{\substack{t \in M \\ t \neq q}} a_{pt} c_t \right) - a_{pq} \left(d_w c_w + \sum_{\substack{t \in M \\ t \neq q}} a_{wt} c_t \right) = 0. \tag{32}$$

We thus modify the remaining entries of A as follows:

$$\begin{aligned} \bar{a}_{wq} &= a_{wq} \\ \bar{a}_{wt} &= a_{wq} a_{pt} - a_{wt} a_{pq} \quad t \in M, \quad t \neq q \end{aligned} \tag{33}$$

Finally, we move p to M_1, q to I_2, and check for an increase in I_1 as above.

Case 20: Retracting a point.

To remove a point from M_1 we move it to M_2 and make any implied moves from I_1 to I_2.

4. A Brief History of Minimal Determining Sets

In this section we give a brief history of research papers that pertain directly to the subject of this paper, i.e., multivariate splines defined on triangulations, the Bernstein-Bézier form, and minimal determining sets. The list does not include literature on other notions of splines, splines on special triangulations, approximation order, subdivision schemes, parametric splines, and a host of other topics. Nor does it include expository or survey articles.

The subject started in 1973 with Strang's paper [25] in which he recognized the difficulty of the dimension problem and offered a conjecture on the dimension of bivariate spline spaces. 1975, in a fundamental paper [20], Morgan and Scott computed the dimension of S^1_d in the case that $d \geq 5$. In 1979, Schumaker, [21], established lower bounds on the dimension of S^r_d, on triangulations and more general rectilinear partitions, for general r and d. In the same paper, he gives the dimension of S^r_d on cells (i.e., vertex stars), for general r and d.

Farin's 1979 thesis, [14], introduced the Bernstein-Bézier form of a bivariate polynomial. In 1984, Schumaker complemented his earlier paper [21] by giving upper bounds on the dimension of spline spaces in [22]. The paper [3] introduces the concept of minimal determining sets (although they are called *annihilating sets* there). It gives minimal determining sets for the case that $d \geq 4r+1$ that are completely specified except in the $2r$ disks around interior vertices of the triangulation. That hole is partially filled in [4] where points in the $2r$ disks are specified for the special case $r = 1, 2, 3$. Also in 1987, the paper [5] constructs a minimal determining set for the special case of S^1_4 on general triangulations.

In 1988, using a completely different approach and sophisticated algebraic machinery, Billera, [11] derived a linear system that describes bivariate splines on generic triangulations in the case $r = 1$. Whiteley, [26], later analyzed this system, using another body of sophisticated (vertex splitting) techniques and established the generic dimension of S^1_d, in particular for the so far still open cases case $d = 2$ and $d = 3$.

In 1988, Schumaker, [23], constructed minimal determining sets for spline spaces on cells, in the process filling in the gaps left in [3] and [4], and establishing the dimension of S^r_d on general triangulations for the case $d \geq 4r + 1$. The 1990 paper [6] gives minimal determining sets for S^r_d in the case $d = 3r + 1$, on almost all triangulations, thereby establishing the generic dimension of these spaces.

In 1991, Hong, [16], constructed minimal determining sets of the case $d \geq 3r + 2$. Ibrahim and Schumaker, [18], modified the approach and generalized it to supersplines (splines with increased smoothness requirements at the vertices of the triangulation).

In 1993, [7], we expand our techniques into spaces of trivariate functions (defined on three-dimensional triangulations) to compute the generic dimension of spline spaces with

$r = 1$ and $d \geq 8$. The basic ideas are to think of trivariate splines paces on vertex stars as the direct sum of bivariate spline spaces on generalized triangulations, and to use vertex splitting to establish the dimension of these spaces. There are fundamental obstacles to extending this analysis to greater values of r or smaller values of d, and the subject is incomplete and very difficult. The 1996 paper [10] gives upper and lower bounds for spline spaces in an arbitrary number of variables and for all values of r and d.

References

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