"Clay Millenium Problems"

Math 1220-3

Notes of 4/11/18

Announcements

- Q&A tomorrow
- New Syllabus online. Next week: more on Taylor Series!

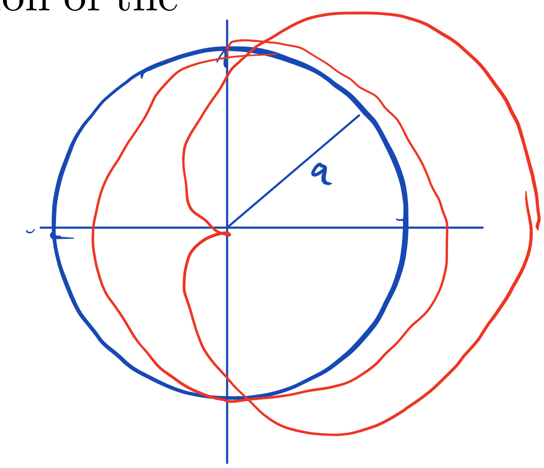
10.6 Polar Equations

- The **graph** of a polar equation is the set of all points $P(r, \theta)$ whose polar coordinates r and θ satisfy the equation.
- Same as before, replace "polar" with "cartesian".
- We already saw some examples. Today, let's start with a gallery of more examples.
- Here are a few more.
- A **limaçon** is the graph of an equation of the form

$$r = a \pm b \cos \theta$$

  where a and b are real numbers.

- A limçacon is a **cardioid** if $a = b$.
- If $b = 0$ a limaçon becomes a circle.



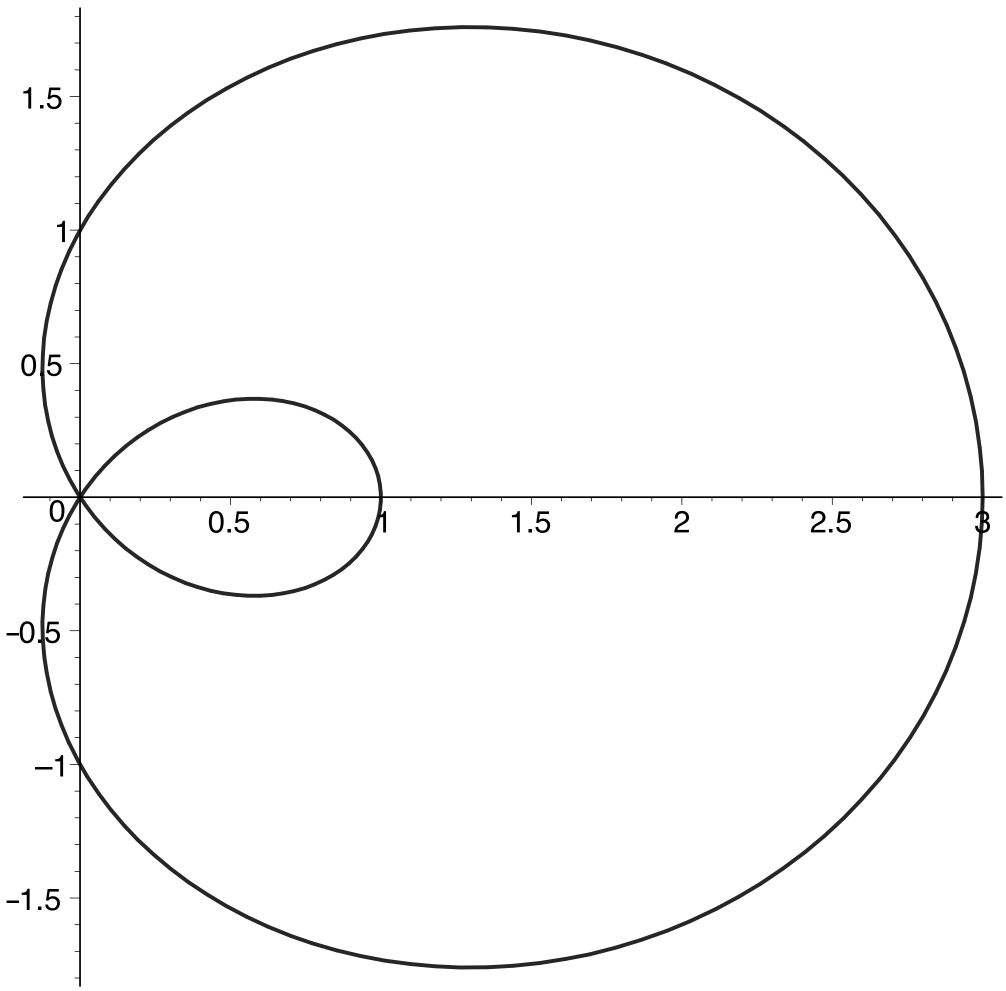


Figure 1. A Limaçon, $r = 1 + 2\cos(t)$.

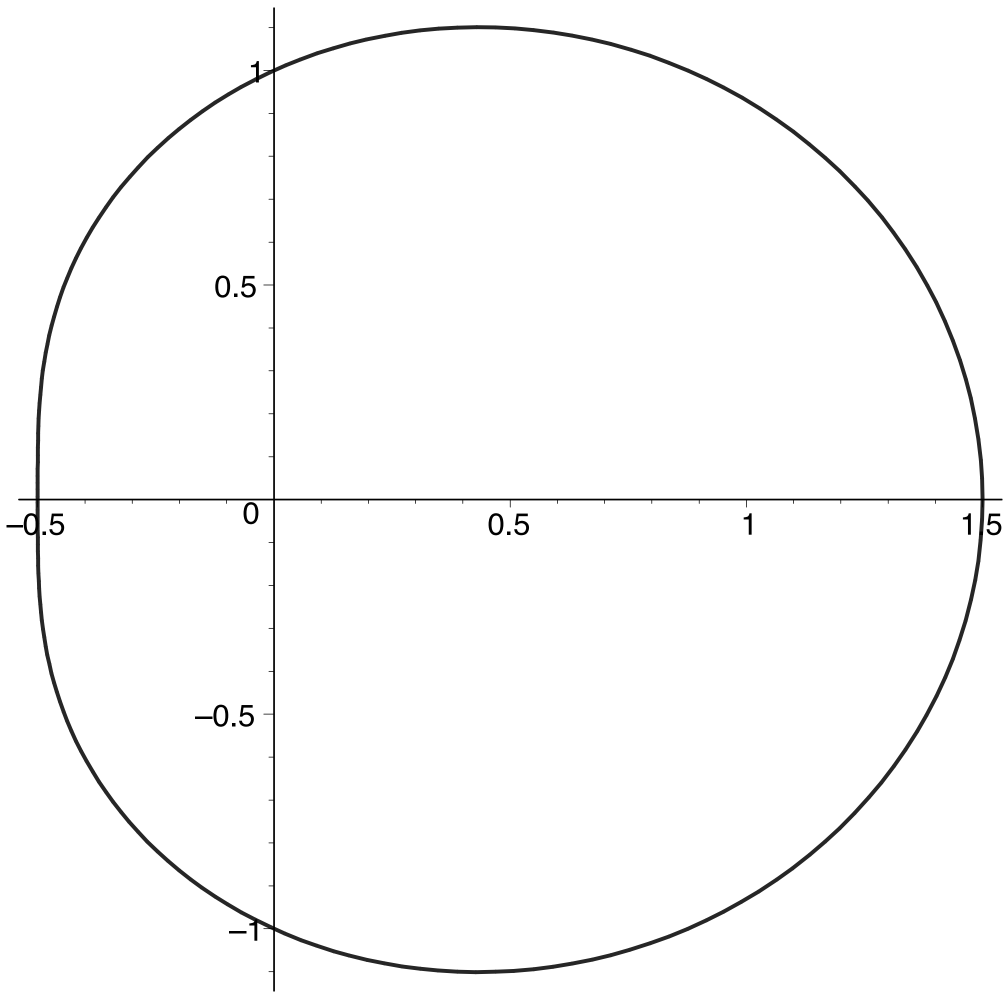


Figure 2. A Limaçon, $r = 1 + 0.5\cos(t)$.

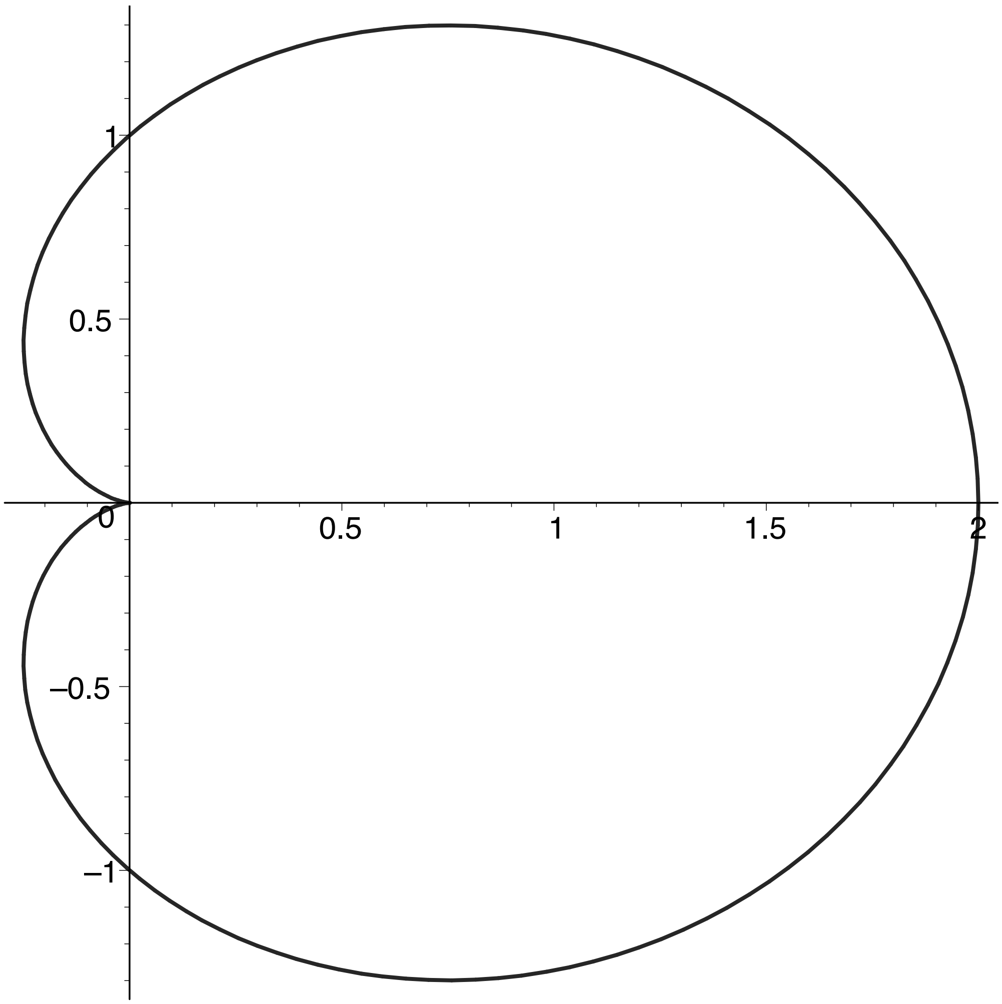
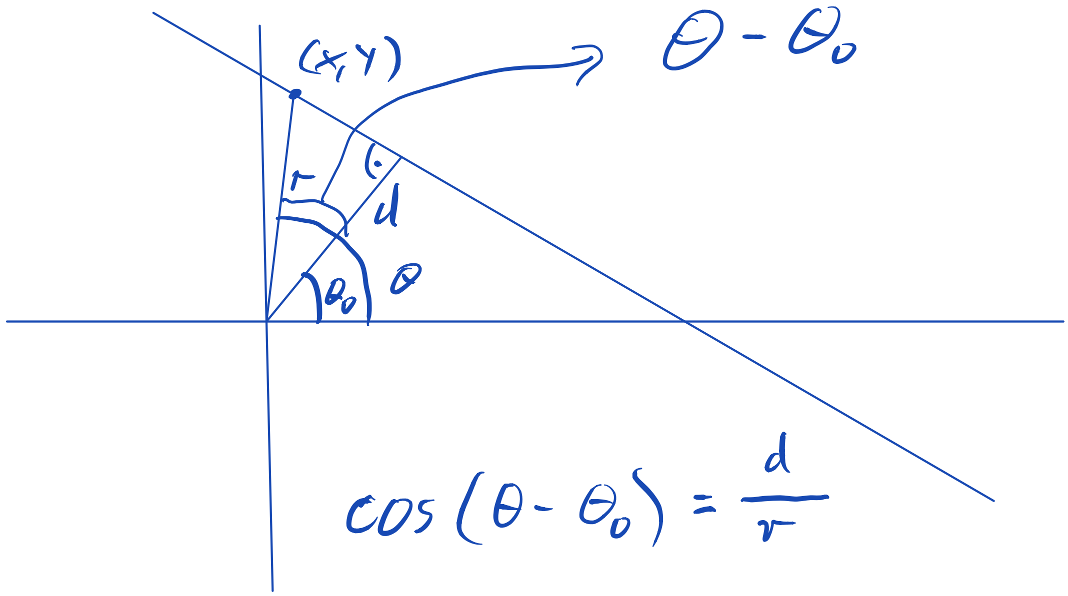


Figure 3. A Cardioid, $r = 1 + \cos(t)$.

Equation of a Line, Revisited

- yesterday we derived the equation of a line in terms of a distance and angle.



$$\cos(\theta - \theta_0) = \frac{d}{r}$$

$$r = \frac{d}{\cos(\theta - \theta_0)}$$

- Here is another form of this equation, more directly related to the standard cartesian form

$$y = mx + b$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r\sin\theta = mr\cos\theta + b$$

$$r(\sin\theta - m\cos\theta) = b$$

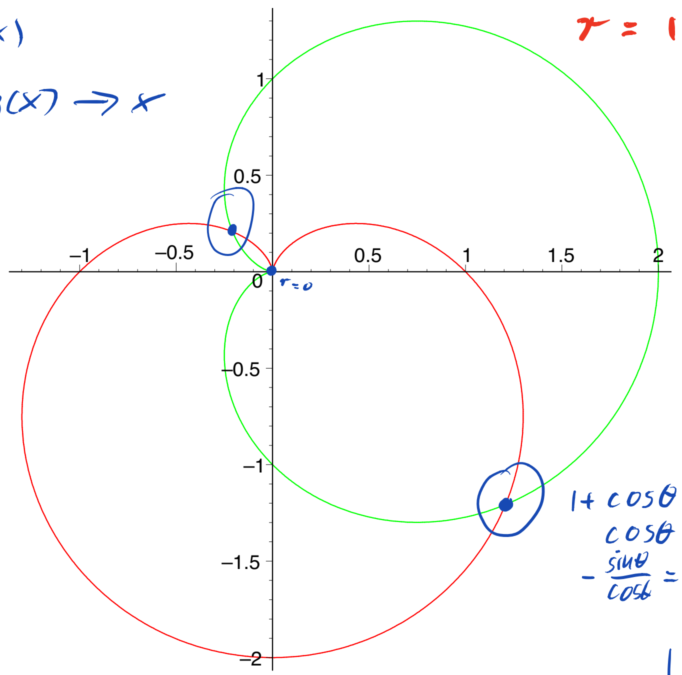
$$r = \frac{b}{\sin\theta - m\cos\theta}$$

Intersections of Polar Curves

- Intersections of curves are more tricky because you can get the same points for different values of the angle.

cartesian
$y = f(x)$
$y = g(x)$
$f(x) = g(x) \Rightarrow x$

$r = 1 + \cos\theta$
$r = 1 - \sin\theta$



$1 + \cos\theta = 1 - \sin\theta$
$\cos\theta = -\sin\theta$
$-\frac{\sin\theta}{\cos\theta} = -\tan\theta = 1$
$\tan\theta = -1$

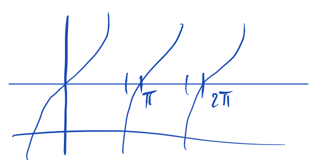


Figure 4. 2 Cardioids.

- Figure 4 shows two cardioids: $r = 1 + \cos\theta$ in green and $r = 1 - \sin\theta$ in red.
- Recall how we computed the the intersection of two curves in Cartesian Coordinates. If these are given by $y = f(x)$ and $y = g(x)$, say, we solve the equation

$$f(x) = g(x)$$

for x and then compute y for those values of x.

- Similarly, the two curves will intersect in a point if r and θ agree. Eliminating r from the two equations gives

$$\begin{aligned} 1+\cos\theta &= 1-\sin\theta \\ \cos\theta &= -\sin\theta \\ \tan\theta &= -1 \end{aligned}$$

which gives the two angles

$$\theta = \frac{3\pi}{4} \quad \text{and} \quad \theta = \frac{7\pi}{4}$$

corresponding to the points

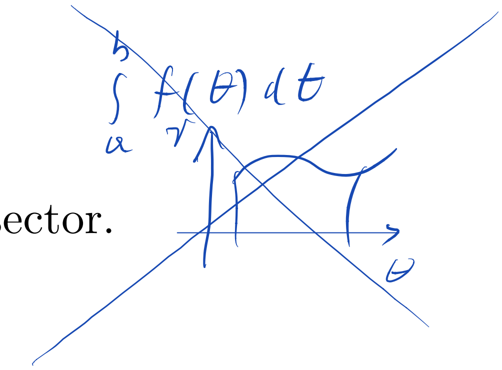
$$P_1 = P\left(1 - \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right) \quad \text{and} \quad P_2 = P\left(1 + \frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right)$$

- Those points are clearly shown in Figure 4.
- However, the two curves also intersect in the origin.
- There $r = 0$ and we get

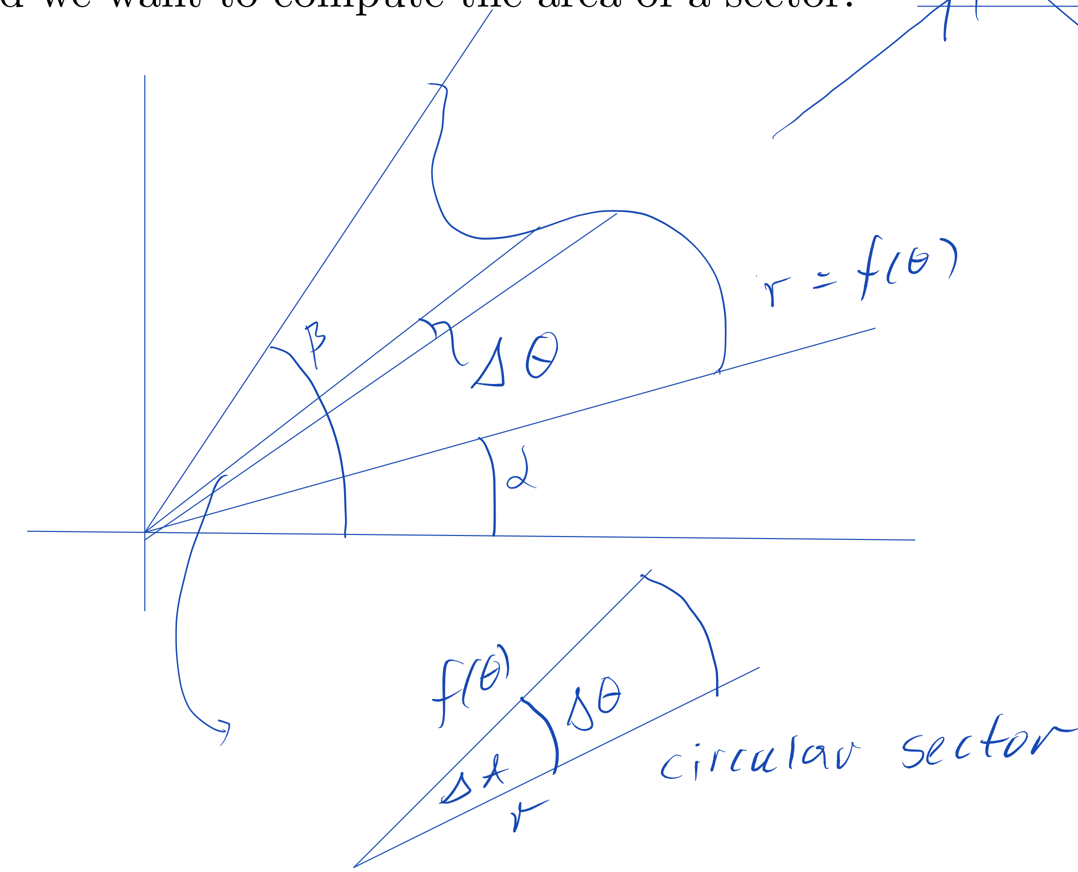
$$\begin{array}{rclcrclcrcl} 1-\sin\theta &=& 0 & \Longrightarrow & \sin\theta &=& 1 & \Longrightarrow & \theta &=& \frac{\pi}{2} \\ 1+\cos\theta &=& 0 & \Longrightarrow & \cos\theta &=& -1 & \Longrightarrow & \theta &=& \pi. \end{array}$$

- The curves pass through the origin for different values of θ. If you think of them just as curves they do of course intersect there. If you think of $\theta = t$ as time and $P(r,t)$ being the location of a vehicle at time t they will coolide in P_1 and P_2, but they will be at the origin at different times, and will not collide there.

10.7 Calculus in Polar Coordinates



- Suppose
$$r = f(\theta)$$
and we want to compute the area of a sector.



$$\Delta A \approx \frac{1}{2} r^2 \Delta\theta$$

$$A \approx \frac{1}{2} \sum f^2(\theta_i)\, \Delta\theta \to \frac{1}{2}\int_\alpha^\beta f^2(\theta)\, d\theta$$

- We get the formula

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\mathrm{d}\theta.}$$

- As usual, let's check it in a case where we know the true answer.
- For the unit circle we get

$$A = \frac{1}{2}\int_{0}^{2\pi} 1^2\mathrm{d}\theta = \pi.$$

- good!
- Example 1, textbook: Compute the area enclosed by the limaçon

$$f(\theta) = 2 + \cos\theta.$$

  shown in Figure 5.
- First estimate!

$A \approx \pi 2^2 = 4\pi$

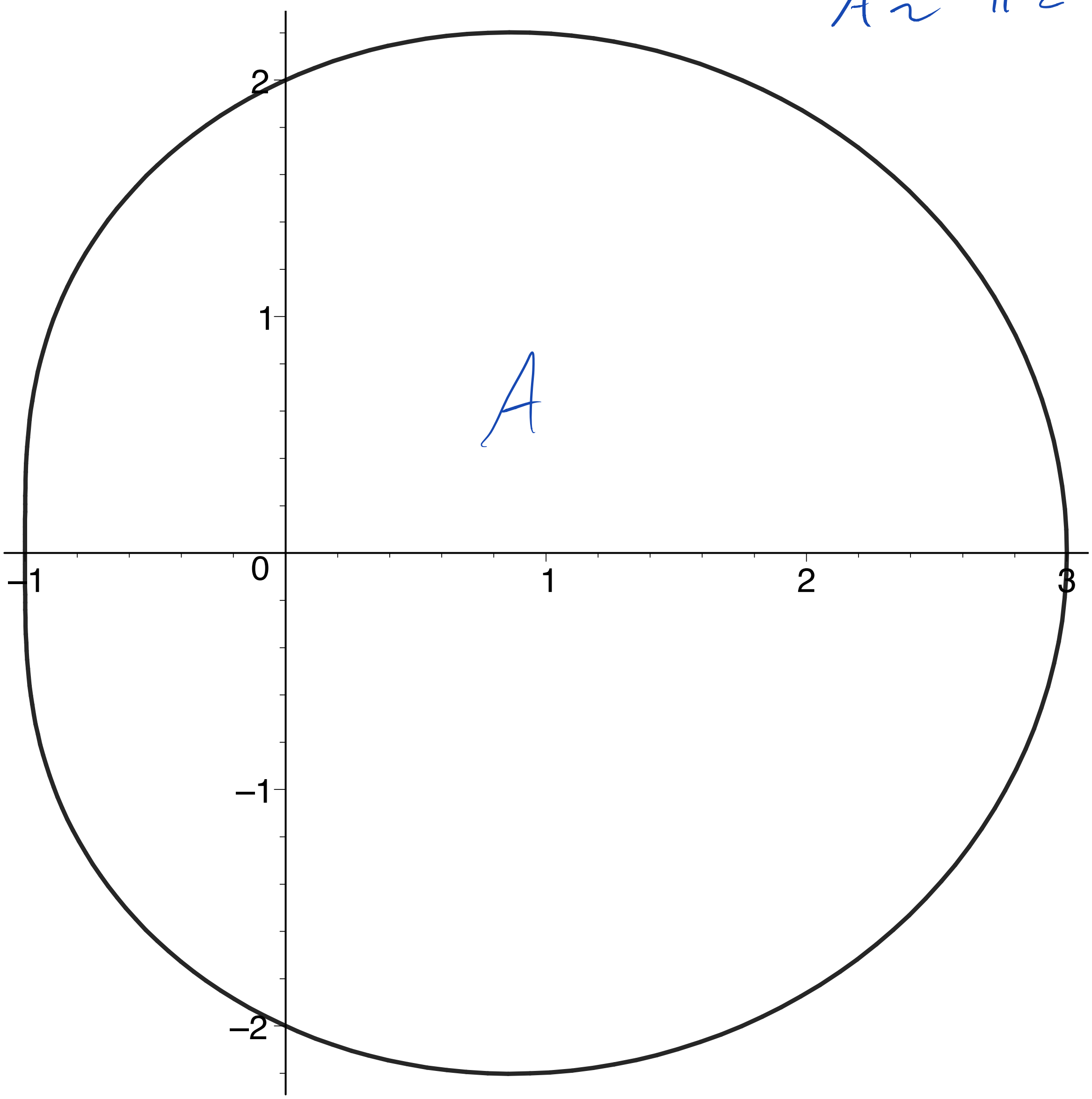


Figure 5. Example 1, Area of Limaçon.

$f(\theta) = 2 + \cos\theta$

$$A = \frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4 + 4\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\left(8\pi + 0 + \pi \right) = \frac{9\pi}{2}$$

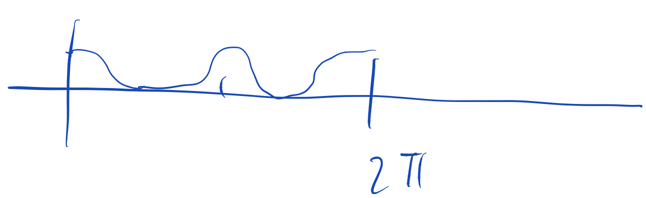
$$\int_0^{2\pi} 4 \, d\theta = 8\pi$$

$$\int_0^{2\pi} 4\cos\theta \, d\theta = 0$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \pi \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} 1 + \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\cos 2\theta\right)\Big|_0^{2\pi} = \pi$$



Compute the area of the circle defined by

$$r = \sin\theta.$$