Math 1220-3, 7, 90

Notes of 8/25/20

Review of some Prerequisites

- This is a review of an entire semester. We will be moving much more slowly (like 40 times as slowly) through the subject of Math 1220.
- If any item does not make sense to you then your should review the relevant material.

These Notes are numbered for easy reference.

1. Prerequisites: know arithmetic, algebra, geometry, cartesian coordinates, functions, trigonometry.

2. Basic functions: polynomials, rational functions, radical functions, trigonometric functions, exponentials, and logarithms.

3. Exponential:

$$f(x) = a^x.$$

$a > 0$ is the **base**, x is the exponent.

4. Logarithms are the inverse of the exponential:

$$a^{\log_a x} = x \quad \text{and} \quad \log_a a^x = x.$$

5. Particularly important bases are $e = 2.71828\ldots$ (natural logarithm and exponential), 2, and 10 (common logarithm).

6. Some Examples:

$$2^3 =$$

$$3^2 =$$

$$3^0 =$$

$$3^{-1} =$$

$$9^{\frac{1}{2}} =$$

$$8^{\frac{1}{3}} =$$

$$8^{\frac{2}{3}} =$$

$$8^{\frac{-2}{3}} =$$

$$\log_{10} 100 =$$

$$\log_2 \frac{1}{2} =$$

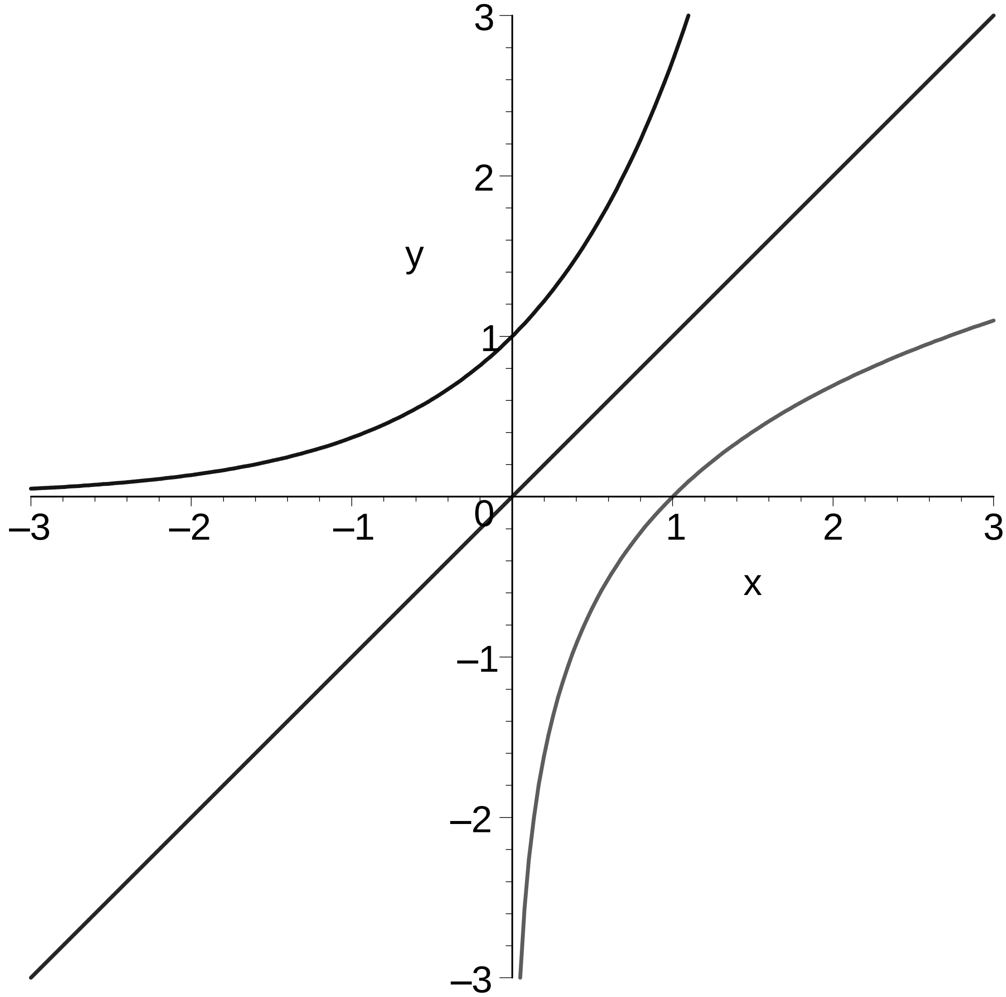

Figure 1. Natural Exponential and Logarithm.

7. Some properties of exponentials and logarithms

$$\boxed{\begin{aligned} a^0 &= 1 \\ a^r a^s &= a^{r+s} \\ \frac{a^r}{a^s} &= a^{r-s} \\ (a^r)^s &= a^{rs} \\ \log(uv) &= \log u + \log v \\ \log \frac{u}{v} &= \log u - \log v \\ \log a^x &= x \log a \\ \log_a x &= \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a} \end{aligned}} \tag{1}$$

8. Know the difference between **simplifying an expression** and **solving an equation**.

9. Example of simplifying an expression:

$$z = \ln(x^2 + 5x + 6) - \ln(x + 3) =$$

$$=$$

10. To solve an equation means to figure out for which values of its variables it is true. We say that those values **satisfy the equation.**

11. Solving equations is a big subject.

12. However, there is just one principle: Apply the same operation on both sides of the equation until you have the variable by itself.

13. How to pick the operations is the crux of the matter, of course, and depends on the context.

Examples

14. How not to solve

$$\frac{x^2}{x+2} = x - 1$$

15. How to solve

$$\frac{x^2}{x+2} = x - 1$$

Always Check Your Answers

16. **Major procedure**: come up with a concept, make it precise, derive its properties, and then use the properties to work with the concept.

17. In Math 1210, we applied this procedure to three major concepts: **limits**, **derivatives**, and **integrals**.

18. Limits. $\lim_{x \longrightarrow c} f(x) = L$ means that for all $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|f(x) - c| < \epsilon$.

19. Most functions we deal with are **continuous**, i.e.,

$$\lim_{x \longrightarrow c} f(x) = f(c) \tag{2}$$

for all c in the domain of f.

20. Intuitively, continuity means that we can draw the graph without lifting the pencil.

21. If we have an expression that is undefined at a point, then manipulate it to get an expression that has the same value as the original everywhere and that can be evaluated at that point. We can do this because of the **Fundamental Limit Theorem** that states facts like that the limit of the sum, difference, product, or quotient is the sum, difference, product, or quotient of the limits.

22. Example:

$$\lim_{h\longrightarrow 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\longrightarrow 0}\frac{x^2+2hx+h^2-x^2}{h}$$
$$= \lim_{h\longrightarrow 0}\frac{2hx+h^2}{h}$$
$$= \lim_{h\longrightarrow 0}(2x+h)$$
$$= 2x. \tag{3}$$

23. The main concept we introduced in Math 1210 was that of a **derivative** of a function f:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}f(x) \\ &= \lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{y\longrightarrow x}\frac{f(y)-f(x)}{y-x} \end{aligned} \tag{4}$$

24. Geometrically: the slope of the tangent is the limit of the slopes of the secants.

25. The derivative tell us how quickly a function is changing at the point where we evaluate the derivative.

26. Locally the derivative approximates the function.

27. To compute derivatives we apply their properties, i.e., **Differentiation Rules**

$$
\begin{array}{rcll}
\dfrac{\mathrm{d}}{\mathrm{d}x} x^r & = & r x^{r-1} & \textbf{Power Rule} \\[2ex]
(f+g)' & = & f' + g' & \textbf{Sum Rule} \\[2ex]
(f-g)' & = & f' - g' & \textbf{Difference Rule} \\[2ex]
(kf)' & = & kf' & \textbf{Constant Multiple Rule} \\[2ex]
\dfrac{\mathrm{d}}{\mathrm{d}x} \sin x & = & \cos x & \textbf{Sine Rule} \\[2ex]
\dfrac{\mathrm{d}}{\mathrm{d}x} \cos x & = & -\sin x & \textbf{Cosine Rule} \\[2ex]
(uv)' & = & u'v + uv' & \textbf{Product Rule} \\[2ex]
\left(\frac{u}{v}\right)' & = & \dfrac{u'v - uv'}{v^2} & \textbf{Quotient Rule} \\[2ex]
\dfrac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) & = & f'\big(g(x)\big) g'(x) & \textbf{Chain Rule}
\end{array}
\tag{5}
$$

28. Differentiation can be repeated, giving rise to higher derivatives, for example:

$$\begin{aligned} f(x) &= x^6, \\ f'(x) &= 6x^5, \\ f''(x) &= 30x^4, \\ f'''(x) &= 120x^3, \\ f^{(4)}(x) &= 360x^2, \\ f^{(5)}(x) &= 720x, \\ f^{(6)}(x) &= 720, \\ f^{(7)}(x) &= 0. \end{aligned} \tag{6}$$

29. Differentiating a polynomial reduces its degree by 1.

30. In general, a function f is a polynomial of degree up to n if and only if the $(n+1)$-th derivative of f is everywhere zero.

31. **Newton's Method** can be used to find a root of a function f, i.e., a solution of the equation $f(x) = 0$. The basic idea is to construct a sequence where each term is the x-intercept to the tangent at the point corresponding to the previous term. You want to understand the formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{7}$$

This means you can derive it and apply it.

32. Differentiation can be done **implicitly**, for example, thinking of y as a function of x, we get

$$x^2+y^2=1 \qquad \Longrightarrow \qquad 2x+2yy'=0 \qquad \Longrightarrow \qquad y'=-\frac{x}{y} \tag{8}$$

33. Implicit Differentiation occurs frequently in **Related Rates Problems**: Write one or more equations that hold at all time, differentiate, obtain equations that involve rates (derivatives), solve for what you want to know.

34. **Differentials:** The change in a function value is approximately equal to the change in the independent variable, multiplied with the derivative.

$$\Delta y \approx \mathrm{d}y = f'(x)\mathrm{d}x = f'(x)\Delta x. \tag{9}$$

35. Source of word problems: the derivative of position is velocity, the derivative of velocity is acceleration.

36. Minima and maxima can occur only at **critical points**:

- **end points** of intervals,

- **singular points** where the derivative does not exist,

- **stationary points**, where the derivative is zero.

37. If $f'(x)=0$ and $f''(x)>0$ then $f(x)$ is a local minimum.

38. If $f'(x)$ is positive (negative) in some interval then f is increasing (decreasing) in that interval.

39. If $f''(x)$ is positive (negative) in some interval then f is concave up (down) in that interval.

40. A **point of inflection** is a point on the graph where the second derivative changes sign.

41. You should be able to draw graphs of functions using many sources of information, for example symmetry, singularities, asymptotes, and first and second derivatives. It's only rarely appropriate simply to compute a large number of points and plot them in a coordinate system.

42. Rational Functions have asymptotes, vertical ones where the denominator is zero, the x-axis if the degree of the denominator exceeds that of the numerator, horizontal ones if the degrees of numerator and denominator are the same, and slanted ones if the degree of the numerator exceeds that of the denominator by 1.

43. The **Mean Value Theorem for derivatives**: If f is differentiable in (a,b) and continuous in $[a,b]$ then there is a point c in (a,b) such that

$$f(b) - f(a) = f'(c)(b-a). \tag{10}$$

44. **Differential Equations**: Equations that involve a function and some of its derivatives. Usually the goal is to find the function.

45. **Antiderivatives**. F is an antiderivative of f if $F' = f$. An integrable function f has infinitely many antiderivatives. Any two antiderivatives differ only by a constant.

46. **Indefinite Integrals**.

$$\int f(x)\mathrm{d}x = F(x) + C \tag{11}$$

where f is the **integrand**, $F' = f$, and C is the **integration constant**. F is an **antiderivative** of f. The indefinite integral is the set of all antiderivatives. The value of the integration constant may be determined by a side condition.

47. Definite Integrals as limits of Riemann Sums:

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\longrightarrow\infty} \sum_{i=1}^n f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \tag{12}$$

48. When computing Riemann Sums, some special sum rules are useful:

$$\begin{aligned} \sum_{i=1}^n 1 &= n, \\ \sum_{i=1}^n i &= \frac{n(n+1)}{2}, \\ \sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned} \qquad . \tag{13}$$

49. If $f(x)$ is non-negative everywhere in $[a, b]$ then $\int_a^b f(x)\mathrm{d}x$ is the area of the region bounded by the x-axis, the graph of f, and the vertical lines $x = a$ and $x = b$.

50. We use Riemann sums to recognize as a definite integer what we are trying to compute. However, we usually compute definite integrals by one version of the **Fundamental Theorem of Calculus:**

$$\boxed{\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F' = f.} \tag{14}$$

51. We usually compute definite integrals by finding an antiderivative. A major topic this semester (in 1220) will be to find ways of doing this, i.e., to find **integration techniques**.

52. In 1210 we discussed (briefly) only one major technique, **integration by substitution**.

53. This is a systematic implementation of the inverse process of the chain rule:

$$\int f'\big(g'(x)\big)\mathrm{d}x = f\big(g(x)\big) + C$$

54. This works, by the fundamental theorem of calculus, since

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'\big(g(x)\big)g'(x).$$

55. Example:

$$I = \int_0^1 \frac{10x}{(x^2+4)^2}\ \mathrm{d}x$$

56. However, some integrals can be computed without knowing an antiderivative. In Math 1210, we discussed in particular:

$$\int_{-c}^{c} f(x)\mathrm{d}x = 0 \quad \text{if } f \text{ is odd,} \tag{15}$$

$$\int_{a}^{a+2\pi} \sin^2 x \mathrm{d}x = \int_{a}^{a+2\pi} \cos^2 x \mathrm{d}x = \pi, \tag{16}$$

and

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\mathrm{d}x = \frac{\pi r^2}{2}. \tag{17}$$

57. On the final exam for Math 1210 it was a common misconception that the integral (17) was zero, because of the symmetry. This can't possibly be true since **the integrand is never negative.**

58. The **Mean Value Theorem for Integrals**: There is a point c in $[a, b]$ such that

$$\int_{a}^{b} f(x)\mathrm{d}x = f(c)(b - a). \tag{18}$$

59. You can be casual when computing an antiderivative, once you have it check it by differentiation.

60. We used definite integrals to solve the following problems:

- Computation of areas

- Computation of volumes by integrating the area of the cross section, using the methods of slabs, disks, washers, or shells.

- Computation of the length of a plane curve.

- Computation of the surface area of a solid of revolution

- Computation of Work.

- Computation of the center of mass.

61. The limits of integration may depend on a variable. We can differentiate with respect to that variable without actually computing an antiderivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{L(x)}^{U(x)} f(t)\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(F\big(U(x)\big) - F\big(L(x)\big)\Big) \\ &= f(U(x))U'(x) - f(L(x))L'(x)\end{aligned} \tag{19}$$

where, as usual, F is any antiderivative of f.

62. A special case of that formula is this version of the Fundamental Theorem of Calculus:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x).} \tag{20}$$

63. The Fundamental Theorem of Calculus says that differentiation and integration are inverse processes of each other.

64. Following is a list of some words and phrases, listed in alphabetical order, that you should be able to define, use, and understand.

acceleration, antiderivative, asymptote, base, chain rule, concave down, concave up, constant, continuity, critical points, cubic, decreasing function, definite integral, degree of a polynomial, denominator, dependent variable, derivative, differential, differential equation, domain, even function, equation, exponent, expression, first derivative, function, Fundamental Limit Theorem, Fundamental Theorem of Calculus, graph (of a function or an equation), implicit differentiation, increasing function, indefinite integral, independent variable, inflection point, integrand, integration constant, integration variable, Leibniz notation, limit, limits of integration (upper and lower), linear, Mean Value Theorem for derivatives, Mean Value Theorem for integrals, method of disks, method of shells, method of slabs, method of washers, Newton's method, numerator, odd functions, points of inflection, polynomial, position, power rule, power, product rule, quadratic, quartic, quintic, quotient rule, radical, range, rational function, related rates, Riemann Sum, secant, second derivative, singular point, solid of revolution, stationary point, sum rules, tangent, velocity, work.

65. **Contents of Math 1220** More differentiation and integration rules (particularly exponentials, logarithms, inverse trig functions, integration by parts, logarithmic differentiation), more applications, indeterminate expressions, improper integrals, sequences and series (particularly their convergence, power series, Taylor series).