Math 1220-3

Notes of 2/5/18

Summary of Chapter 6

As you know, our first midterm exam will take place Friday, February 9, at our regular class time. It will be on Chapter 6. Note, however, that integration by substitution, which is the focus of section 7.1, actually already occurred in Math 1210, way back in section 3.8, and thus is a familiar technique. You may find integration by substitution appropriate for one or two problems on the exam.

Since this is our first exam, here is a transcript of the instructions on the cover page of the exam:

Instructions

0. Before you do anything else write your name on the top of this page.

1. This exam is closed books and notes, no electronic devices of any kind.

2. Use these sheets to record your work and your results. Use the space provided, and the back of these pages if necessary. **Show all work.** Unless it's obvious, indicate the problem each piece of work corresponds to, and for each problem indicate where to find the corresponding work.

3. Please note: **To avoid disruption and distraction I won't be able to answer questions during the exam. If you believe there is a mistake in one of the problems write down an appropriate note and if you are right you will receive generous credit.**

4. The questions on this test are deliberately simple. You should not be rushed and have time to answer all questions carefully and check your answers. **Accuracy is more important than speed.** Don't get stuck on one problem. If you can't answer a question immediately go on and return to that question only after you have answered the others.

5. **Simplify any algebraic expressions and reduce any fractions. You don't need to approximate mathematical expressions with decimal expressions. Do not omit the integration constant for indefinite integrals.**

6. If you are done before the allotted time is up I recommend strongly that you stay and use the remaining time to check your answers. In particular, check your integrals by differentiation.

7. All questions have equal weight.

Write your final answers (and nothing else) in the boxes provided.

8. On your way out pick up an answer sheet. Do not return to your seat.

Be Prepared

- The exam is focused on the manipulation, properties, derivatives, and integrals of functions involving exponentials and logarithms, and inverse trig functions. The key to doing well on any exam is to understand the subject. Focusing on a narrow list of anticipated questions is counterproductive. However, as discussed at the beginning of the semester, the exam questions will be taken straight from the home works (hw 1–4) and the class notes. I may simplify the problem a bit, take only part of a problem, and change some numbers. (Many ww problems have different numbers for different people anyway.)

You want to be done with home work 4 before the exam, even though the hw closes only later that day.

To be well prepared for the exam you want to understand how you computed the answers to the home work problems. It's not enough just to have gotten credit for your answer. You also want to understand the online notes and what we did in class throughout the beginning of this semester.

Topics

- Following is a review of the subject we covered in chapter 6. The list is neither self contained nor complete. Each item should trigger your memory and understanding of the surrounding subject. If it does not then go back over your notes, the textbook, and past home work problems, and make sure it does.
- Definition of the natural logarithm:

$$\ln x = \int_1^x \frac{1}{t} \mathrm{d}t \tag{1}$$

- Properties of the natural logarithm (same as for any logarithm)

$$\begin{aligned} \ln(uv) &= \ln u + \ln v, \\ \ln\left(\frac{u}{v}\right) &= \ln u - \ln v, \\ \ln(1) &= 0, \\ \ln\left(u^v\right) &= v \ln u. \end{aligned} \tag{2}$$

- The derivative of the natural logarithm, by its definition and the fundamental theorem of calculus:

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} \tag{3}$$

- The natural exponential is the inverse of the natural logarithm:

$$\begin{aligned} \exp(\ln x) &= x, & x > 0 \\ \ln \exp(x) &= x, & -\infty < x < \infty \end{aligned} \tag{4}$$

- The natural exponential is the familiar exponential with base e.

$$\exp(x) = e^x, \qquad e = 2.71828\ldots \tag{5}$$

- Be clear in your mind about domain and range of exponential and logarithmic functions.
- The exponential equals its own derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x. \tag{6}$$

- It has the standard properties:

$$e^{u+v} = e^u e^v, \qquad (e^u)^v = e^{uv}, \qquad e^0 = 1. \tag{7}$$

- Of course you should be able to do anything with exponentials and logarithms that you learned in preCalculus. This includes simplifying logarithmic and exponential expressions, and solving logarithmic or exponential equations.
- Doubling time, half life. You should be able to construct exponential functions with given

initial population, and given half lives or doubling times. You should also be able to compute the time at which the population is multiplied with a given factor.

- Integration by substitution is the inverse process of the chain rule. Thus, if $F' = f$ then

$$\int f\big(g(x)\big)g'(x)\mathrm{d}x = F\big(g(x)\big) + C. \tag{8}$$

  You may, or may not, use the explicit substitution $u = g(x)$. For indefinite integrals, we usually go back to an expression in x, for definitive integrals we do not, but then we have to change the limits of integration. We have seen many examples where the substitution in (8) is obvious, but sometimes it is not.

- Logarithmic differentiation. Take the logarithm first, then differentiate implicitly and solve for the derivative of interest.

- Differentiation of inverse functions: Start with

$$f\big(f^{-1}(x)\big) = x, \tag{9}$$

  differentiate implicitly, and solve for the derivative of f^{-1}. It's better to use and understand the process, but you get the formula

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} \qquad \text{where} \qquad y = f(x). \tag{10}$$

- In particular we applied this idea to obtain the derivative of the exponential, and the derivatives of some inverse trig functions. As mentioned, the exponential equals its own derivative. The most important formula for the inverse trig functions is

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}. \tag{11}$$

- We also saw that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x &= \frac{1}{\sqrt{1-x^2}}\\ \frac{\mathrm{d}}{\mathrm{d}x}\arccos x &= \frac{-1}{\sqrt{1-x^2}} = -\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x\end{aligned} \tag{12}$$

- The hyperbolic sine, cosine, and tangent are defined by

$$\begin{aligned}\sinh t &= \frac{e^t - e^{-t}}{2},\\ \cosh t &= \frac{e^t + e^{-t}}{2}, \quad \text{and}\\ \tanh t &= \frac{\sinh t}{\cosh t}.\end{aligned} \tag{13}$$

- They are called *hyperbolic* because

$$\cosh^2 t - \sinh^2 t = 1 \tag{14}$$

and the graph of $x^2 - y^2 = 1$ is a hyperbola. They are called *sine* and *cosine* because (14) is similar to the fundamental property

$$\cos^2 t + \sin^2 t = 1 \tag{15}$$

of the trigonometric functions.

- Their derivatives are given by

$$\begin{aligned} D_t \sinh t &= \cosh t, \\ D_t \cosh t &= \sinh t, \\ D_t \tanh t &= \frac{1}{\cosh^2 t}. \end{aligned} \tag{16}$$

- We also computed expressions for the inverse of the hyperbolic sine and its derivative.

- Separable differential equations are of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y). \tag{17}$$

  Separate the variables, integrate on both sides, and solve for the function you want.

- First order linear differential equations are of the form

$$y' + P(x)y = Q(x). \tag{18}$$

  The integrating factor

$$e^{\int P(x)\mathrm{d}x} \tag{19}$$

is the exponential of an antiderivative of the factor of the dependent variable. Multiply with the integrating factor, integrate on both sides, and solve for the function you want.

- Differential equations are important because they can be used to model natural processes.
- Most differential equations cannot be solved analytically, and need to be solved approximately. We discussed just one numerical method, Euler's Method, which is the tip of a very large ice-berg.