Math 1220-3

Notes of 1/30/18

6.9 Hyperbolic Functions and their Inverses

- The hyperbolic sine, cosine, and tangent, are defined by

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$

- You might say these are just combinations of exponentials, so they don't deserve treatment as a separate subject.
- However, they come up a lot in applications, and they have a large number of useful properties. We'll look at some of them.

sin, cos $\qquad y'' - y$

Graphs

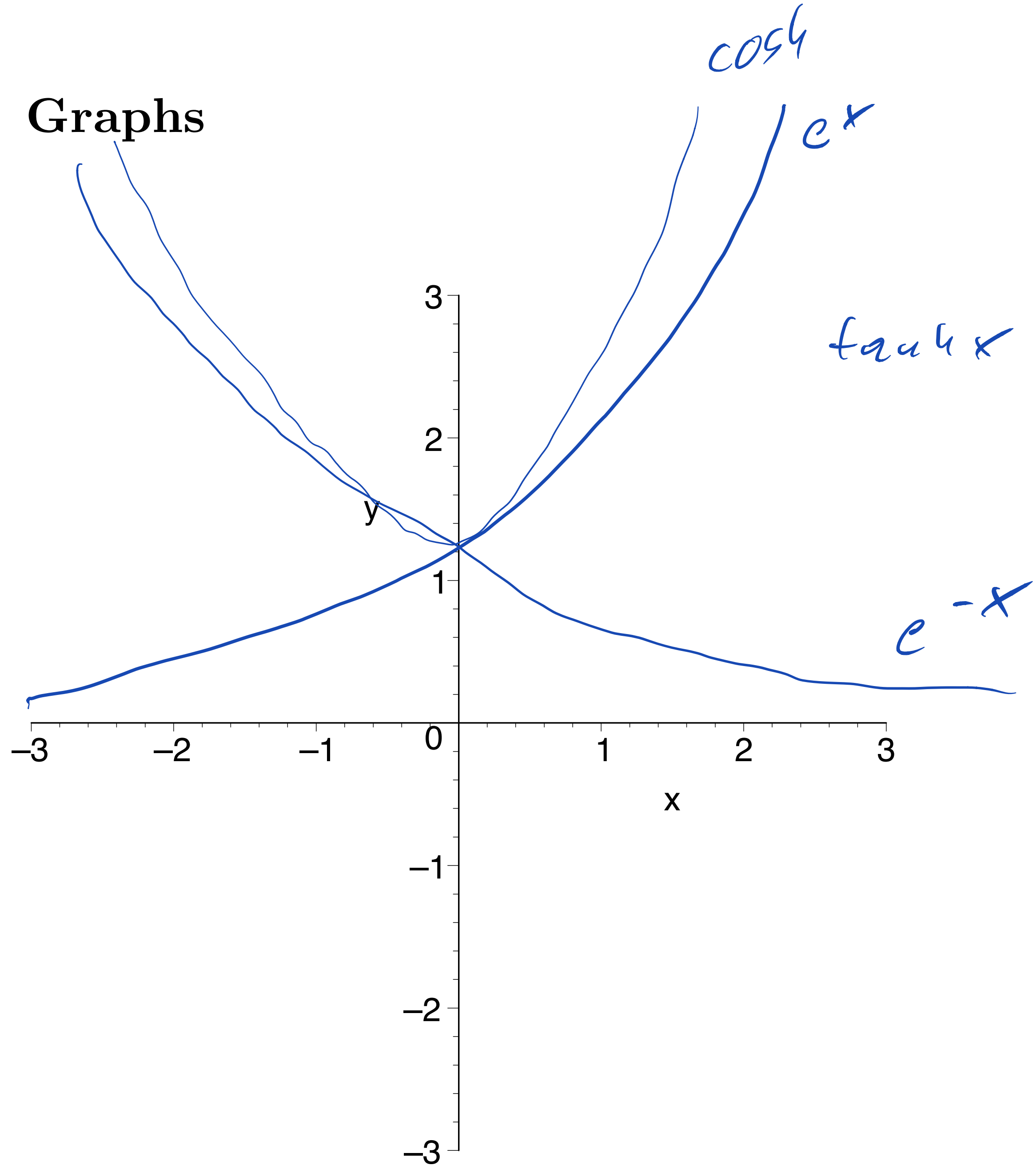


Figure 1. Graphs of hyperbolic functions.

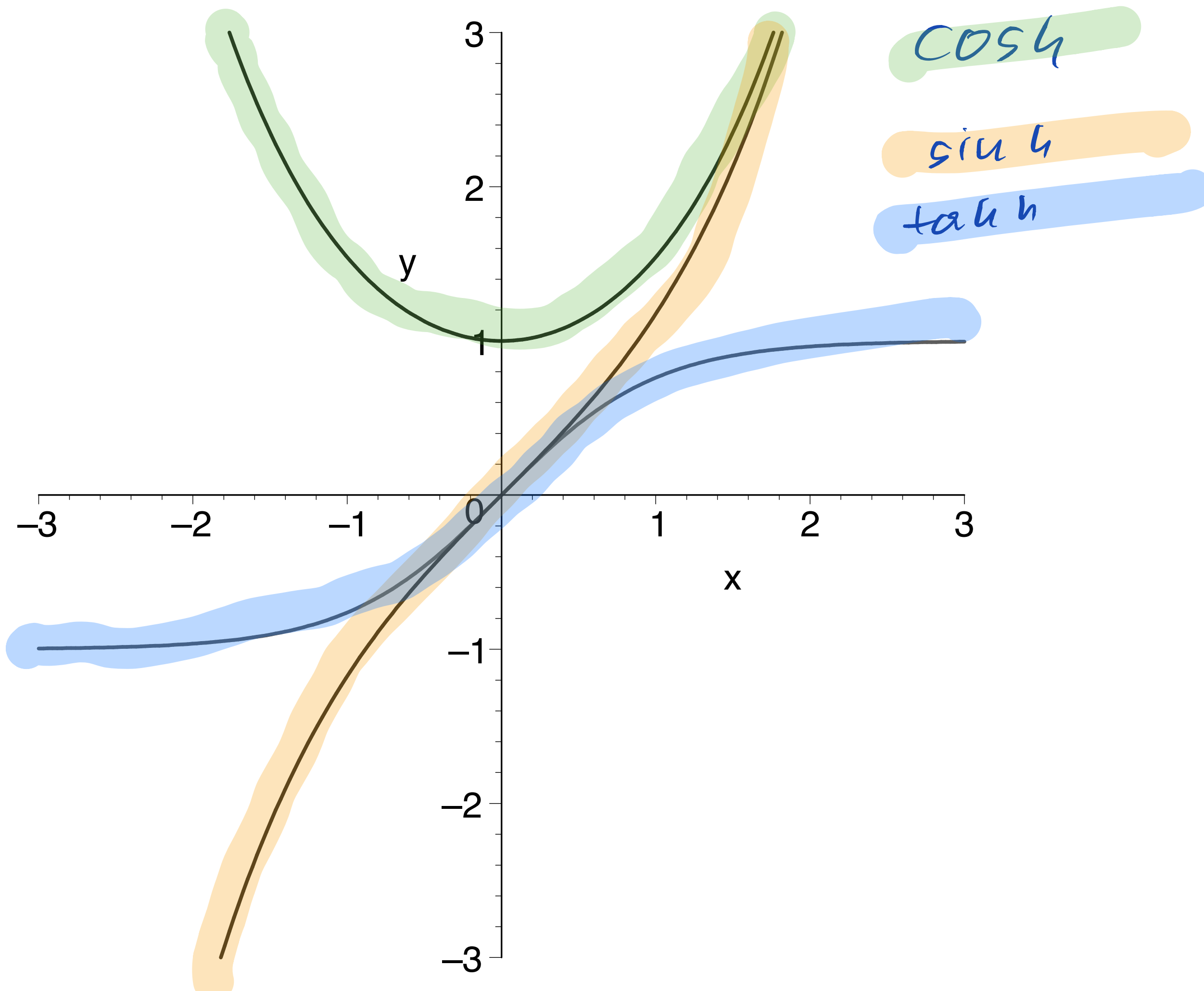


Figure 2. Graphs of hyperbolic functions.

Why "Hyperbolic"?

$$\cosh^2 x = \left(\frac{1}{2}(e^x + e^{-x})\right)^2$$

$$= \frac{1}{4}\left((e^x)^2 + 2(e^x)(e^{-x}) + (e^{-x})^2\right)$$

$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$$

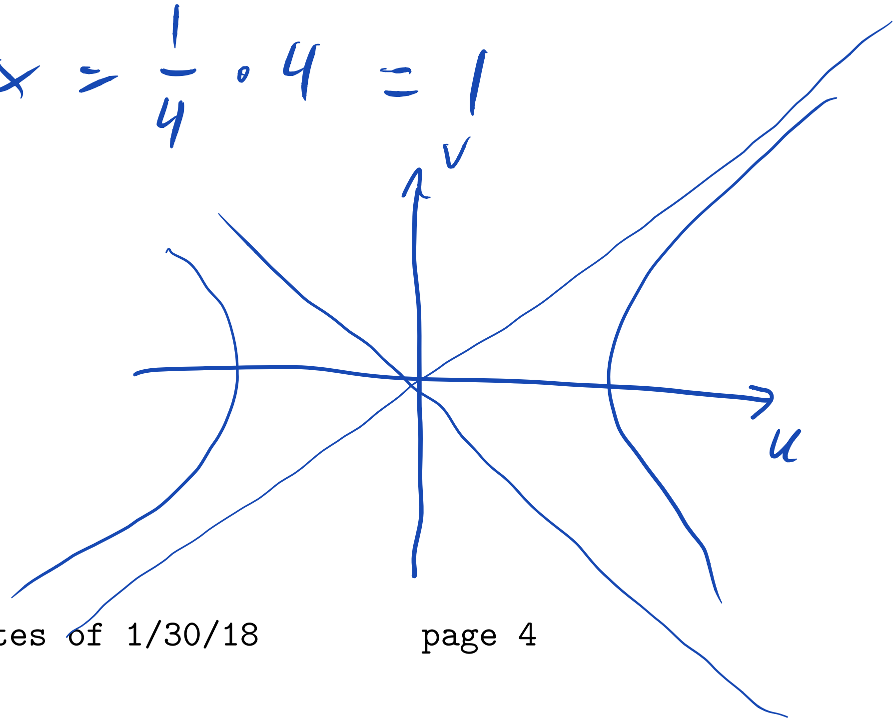
$$\sinh^2 x = \left(\frac{1}{2}(e^x - e^{-x})\right)^2$$

$$= \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}\cdot 4 = 1$$

$$u^2 - v^2 = 1$$

$$u^2 = 1 + v^2$$



Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \frac{d}{dx}\frac{1}{2}\left(e^{x}-e^{-x}\right)$$

$$= \frac{1}{2}\left(e^{x}+e^{-x}\right)$$

$$= \cosh x$$

$y'' = y$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \frac{d}{dx}\frac{1}{2}\left(e^{x}+e^{-x}\right)$$

$$= \frac{1}{2}\left(e^{x}-e^{-x}\right)$$

$$= \sinh x$$

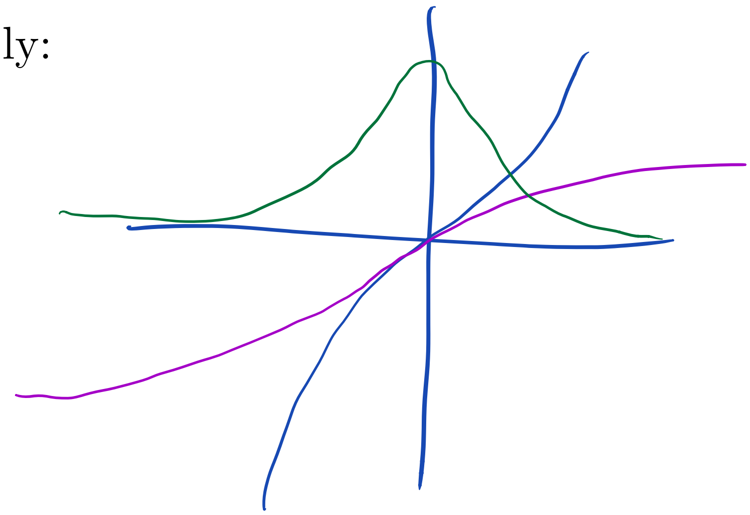
$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x}$$

Derivatives of Inverses

- cosh is not invertible.
- Let's do sinh.
- Differentiate implicitly:



sinh

arcsinh = $\sinh^{-1}$

$\operatorname{arcsinh}'(x)$

$$\sinh \operatorname{arcsinh} x = x \quad \Big| \frac{d}{dx}$$

$$\left(\cosh(\operatorname{arcsinh} x)\right) \cdot \operatorname{arcsinh}'(x) = 1$$

$$\operatorname{arcsinh}'(x) = \frac{1}{\cosh(\operatorname{arcsinh} x)}$$

$$= \frac{1}{\sqrt{1 + \sinh^2(\operatorname{arcsinh} x)}}$$

$$=$$

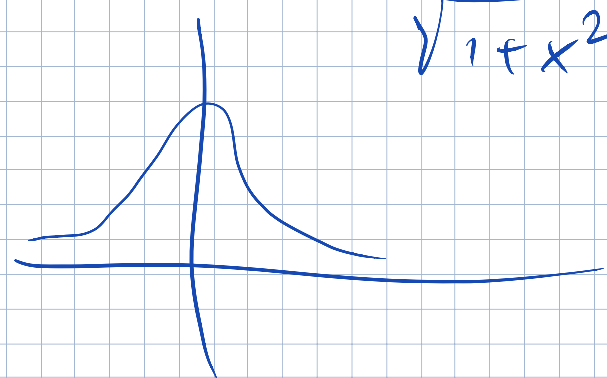
$$1 = \cosh^2 z - \sinh^2 z$$

$$\cosh^2 z = 1 + \sinh^2 z$$

$$\cosh z = \sqrt{1 + \sinh^2 z}$$

$$= \frac{1}{\sqrt{1 + \left(\sinh(\operatorname{arcsinh} x)\right)^2}}$$

$$= \frac{1}{\sqrt{1 + x^2}}$$



- We can compute an explicit expression for $\sinh^{-1}$:

$$y = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$z = e^{x} = ?$$

$$y = \frac{1}{2}\left(z - \frac{1}{z}\right) \qquad | \cdot z \cdot 2$$

$$2yz = z^2 - 1$$

$$z^2 - 2yz - 1 = 0$$

$$z \left(= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$$= \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$

$$= y + \sqrt{y^2+1} \quad = e^{x} > 0 \quad = z$$

$$x = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$$

- and differentiate it explicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x+\sqrt{1+x^2}\right) = \frac{1+\frac{1}{2}\frac{2x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} \cdot \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1+x^2}+x}{x\sqrt{1+x^2}+1+x^2}$$

$$= \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)}$$

$$= \frac{1}{\sqrt{1+x^2}} \quad \text{good}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

A Catenary

- A hanging rope suspended at $x = -a$ and $x = a$ assumes the shape of the **catenary**

$$y(x) = a \cosh \frac{x}{a}$$

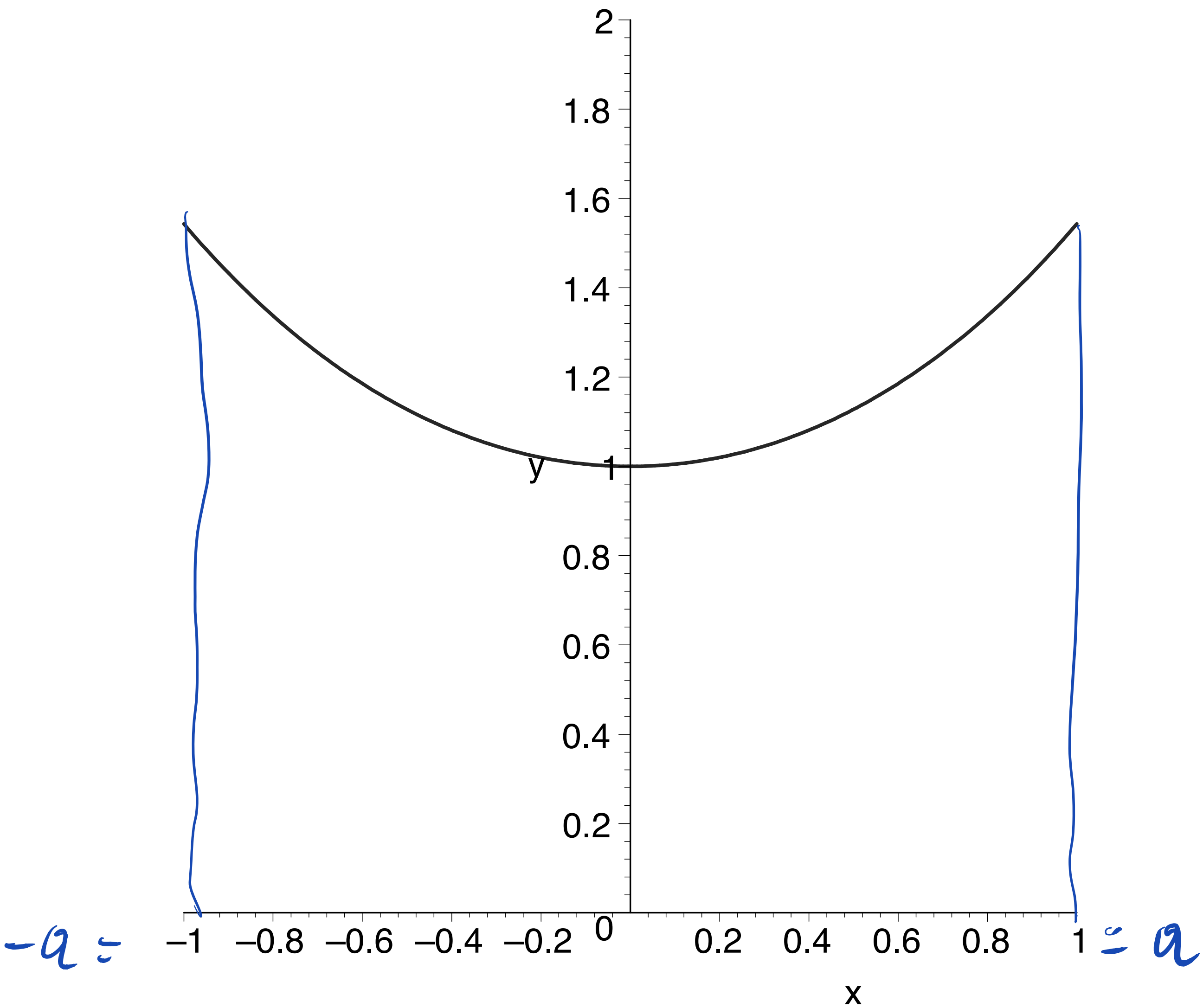


Figure 3. Catenary with $a = 1$.

- Find the length of the catenary

$$y(x) = a\cosh\frac{x}{a}, \qquad -a \le x \le a.$$

- Expectations?

$$L = \int_{-a}^{a} \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_{-a}^{a} \sqrt{1 + \left(\frac{a}{a}\sinh\frac{x}{a}\right)^2}$$

$$= \int_{-a}^{a} \sqrt{1 + \sinh^2\frac{x}{a}}$$

$$= \int_{-a}^{a} \cosh\frac{x}{a}\, dx$$

$$= a\sinh\frac{x}{a}\Big|_{-a}^{a}$$

$$= a(\sinh 1 - \sinh(-1))$$

$$= 2a \sinh 1$$

$$\approx 2.35a$$