Math 1220-3

Notes of 1/26/18

More on DEs

- Since DEs are so important, today let's talk some more about them rather than just going on in the textbook.

- There are three kinds of DEs that we know how to solve analytically:

  - **Integrals:**

  $$y' = f(x), \qquad y(a) = 0, \qquad y(x) = \int_a^x f(t)\mathrm{d}t.$$

  - **Separable DEs** (discussed in 1210). Separate variables and integrate.

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{N(x)}{D(y)} \qquad \longrightarrow \qquad D(y)\mathrm{d}y = N(x)\mathrm{d}x.$$

  - **First oder linear DE:**

  $$y' + P(x)y = Q(x).$$

  Multiply with $e^{\int P(x)\mathrm{d}x}$ and integrate.

  A first order linear DE is **separable** if $Q(x) = 0$ and an **ordinary integral** if $P(x) = 0$.

- The basic philosophy of **solving** DEs is that the DE is "solved" if we can express the answer as an ordinary integral. We have reduced the DE problem to a simpler problem.

A Real Life Problem

- I actually did this!

- In my backyard there is a fish pond containing 2000 gallons of water. Once a week I replace some of the water to reduce the pollutants (e.g., fish waste) that have accumulated in the pond, by pouring water into the pond, and letting the pond overflow. My garden hose delivers 10 gallons of water per minute. The contents of the pond are well mixed. I can't let the water run too long because water from the Salt Lake City main contains traces of chlorine. If all, or much, of the water in the pond was tap water the fish would suffer or even die. How long do I need to let the water run so that 20 percent of the pollutants are removed from the pond?

- Expectations?

> 40 min

10 gal   2000 gal   10 gal

$P(t)$ amount pollutants
at time t

$$P'(t) = \frac{-P \cdot 10}{2000}$$

$$P' = -\frac{1}{200} P$$

$$P(t) = C\, e^{-\frac{t}{200}}$$

$$= P(0)\, e^{-t/200}$$

$$e^{-t/200} = 0.8 \quad \big| \ln$$

$$-t/200 = \ln 0.8$$

$$t = -200 \cdot \ln 0.8$$

$$\approx 46 \text{ min}$$

musings

$$P' = P$$
$$P(t) = C e^{t}$$

$$P' = kP$$
$$e^{\frac{1}{k}t}$$

$$\frac{d}{dt} e^{1/kt} =$$
$$\frac{1}{k} e^{1/kt}$$

$$P' = kP$$
$$P(t) = C e^{kt}$$

$$\frac{d}{dt} e^{kt} = k e^{kt}$$
$$= k P(t)$$

$$2^{t} = \left(e^{\ln 2}\right)^{t}$$
$$= e^{(\ln 2)t}$$
$$k = \ln 2$$

$$2^{t+1} = 2^t \cdot 2^1 = 2 \cdot 2^t$$

- Let's go back to falling objects, with or without air resistance.

No Air Resistance

- This is a review!

- We have three functions:

a acceleration

v velocity

h height

- We also know that

$$a = v' = h'' \quad \text{and} \quad v = h'$$

- On earth, ignoring air resistance, i.e., for small values of v, we have

$$
\begin{aligned}
a &= -32 \quad \text{ft/sec}^2 \\
v &= -32t + v_0 \quad \text{ft/sec,} & v_0 &= v(0) \\
h &= -16t^2 + v_0 t + h_0 \quad \text{ft,} & h_0 &= h(0)
\end{aligned}
$$

- In general, with gravity being denoted by g,

$$
\begin{aligned}
a &= -g \\
v &= -gt + v_0 \\
h &= -\frac{g}{2}t^2 + v_0 t + h_0
\end{aligned}
$$

Air Drag

- But you can't ignore air resistance (or drag)!

- Let's look at the simple case that drag is proportional to velocity. (This is unrealistic, but easy to analyze.)

- So we get the DE

$$v' = -g - kv$$

  where $k > 0$ is the **drag coefficient**.

- Setting $v' = 0$ and solving for v gives the **terminal velocity**

$$0 = -g - kv_\infty \qquad \Longrightarrow \qquad v_\infty = -\frac{g}{k}.$$

- So what is a good value for k?

Googling "terminal velocity of a falling human body" gives

$$v_\infty = -200\text{km/h} \approx -183 \quad \text{ft/sec}.$$

- This gives

$$k = -\frac{g}{v} = \frac{32}{183} \approx 0.17\frac{\text{ft/sec}^2}{\text{ft/sec}} = 0.17\frac{1}{\text{sec}}.$$

- For the sake of curiosity, and the exercise, and maybe the fun of it, let's compute how long it would take a human body to reach the ground from a height of 5 miles, say, with and without accounting for air resistance.

- 5 miles $= 5 \times 5280$ feet $= 26{,}400$ feet.

- Expectations?

No Air Resistance

$$h(t) = -16t^2 + v_0 t + h_0$$

$$v_0 = 0$$
$$h_0 = 26,400$$

$$= -16t^2 + 26,400 = 0$$

$$t = \sqrt{\frac{26,400}{16}}$$

$$v' + P(t)v = Q(t)$$

with Air resistance

$$v' = -g - kv$$

$$v' + kv = -g \qquad |\cdot e^{kt}$$

$$e^{kt}v' + ke^{kt}v = -ge^{kt} \qquad |\int$$

$$e^{kt}v = -\frac{g}{k}e^{kt} + C$$

$$t = 0 \qquad v = 0$$

$$0 = -\frac{g}{k} + C \qquad C = \frac{g}{k}$$

$$e^{kt}v = -\frac{g}{k}e^{kt} + \frac{g}{k} \qquad |\cdot e^{-kt}$$

$$v = -\frac{g}{k} + \frac{g}{k}e^{-kt}$$

$$= -\frac{g}{k}\left(1 - e^{-kt}\right)$$

$$h(t) = \int v(t)dt = -\frac{g}{k}\left(t + \frac{1}{k}e^{-kt}\right) + C'$$

$$h(0) = h_o = 26{,}400 = -\frac{g}{k^2} + q$$

$$q = h_o + \frac{g}{k^2}$$

$$h(t) = -\frac{g}{k}\left(t + \frac{1}{k}e^{-kt}\right) + h_o + \frac{g}{k^2} = 0$$

$$-\frac{32}{0.17}\left(t + \frac{1}{0.17}e^{-0.17t}\right) + 26{,}400 t - \frac{32}{0.17^2} = 0$$

Newton's Method

146 seconds

Falling, Falling, Falling ...

- When my daughter was taking Calculus she asked me how long it would take for the Earth to fall into the Sun if it was to stop dead in its orbit. At the time I was unable to solve the problem analytically and so came up with an answer numerically. My daughter told me that this was not acceptable, she'd like an explicit expression! At the time I felt quite challenged, but it took me about two years to figure out an analytical answer. (I did do some other things during that time as well.) These notes contain the analytical solution. When I told my daughter that I had finally the answered her question she made sure I understood that she was no longer interested ...

- Let's investigate this question more generally. So suppose we have a star or planet $\mathcal{O}$ that has a radius R and a surface gravity G (measured in the appropriate units). For example, the radius of the Sun is 6.96×10^8 meters, and gravity on the Sun's surface is 274 meters per second squared. The (average) distance of the Earth from the Sun is 1.49×10^{11} meters.

- Let's suppose we release a stationary object $\mathcal{E}$ at a distance H from $\mathcal{O}$, and we wish to know how long it will take to fall until it reaches a specified distance from $\mathcal{O}$. We assume $\mathcal{E}$ to be a point, i.e., we ignore its radius. Let $s(t)$ be the distance between $\mathcal{E}$ and $\mathcal{O}$ at time t. s

is the solution of the initial value problem

$$s'' = -G \left(\frac{R}{s} \right)^2, \qquad s(0) = H, \quad s'(0) = 0.$$

(1)

- Multiplying with s' on both sides of the differential equation gives

$$s'' s' = -GR^2 \frac{s'}{s^2}.$$

(2)

- Integrating on both sides gives:

$$\frac{(s')^2}{2} = \frac{GR^2}{s} + C$$

(3)

To find the integration constant C we substitute $s = H$ and $s' = 0$ to get $C = -\frac{GR^2}{H}$. Thus

$$\frac{1}{2}(s')^2 = GR^2 \left(\frac{1}{s} - \frac{1}{H} \right)$$

(4)

and

$$s' = -\sqrt{2G}R\sqrt{\frac{1}{s} - \frac{1}{H}}.$$

(5)

The minus sign is due to the fact that the distance from $\mathcal{O}$ is decreasing as time is increasing. The equation (5) is a *separable differential equation*, giving

$$dt = -\frac{ds}{\sqrt{2G}R\sqrt{\frac{1}{s} - \frac{1}{H}}}.$$

(6)

- Integrating on both sides gives

$$t = -\frac{\sqrt{2}\sqrt{H}\left(-2\sqrt{s}\sqrt{H-s} + H\arctan\left(\frac{2s-H}{2\sqrt{s}\sqrt{H-s}}\right)\right)}{4R\sqrt{G}} + T$$

(7)

where T is an integration constant.

- This result was first obtained with Maple, and subsequently simplified. You can check by differentiation that it is correct.

- Usually one would want s as a function of t, but actually, for our application, having time expressed as a function of distance is exactly what we need.

- We now need to compute the integration constant T. We have $H \geq s$. Hence the argument of the inverse tangent function in (7) approaches positive infinity as s approaches H. The inverse tangent itself approaches $\frac{\pi}{2}$. Taking the limit as s approaches H and t approaches 0 gives

$$T = \frac{\sqrt{2}\pi H^{\frac{3}{2}}}{8R\sqrt{G}}.$$

(8)

- Hence the time required for $\mathcal{E}$ to reach a distance s from $\mathcal{O}$ is

$$t(s) = \frac{\sqrt{2}\pi H^{\frac{3}{2}}}{8R\sqrt{G}} - \frac{\sqrt{2}\sqrt{H}\left(-2\sqrt{s}\sqrt{H-s} + H\arctan\left(\frac{2s-H}{2\sqrt{s}\sqrt{H-s}}\right)\right)}{4R\sqrt{G}}$$

(9)

- Substituting the values

$$H = 1.49 \times 10^{11}, \qquad R = 0.696 \times 10^9, \quad \text{and} \quad s = R$$
<div align="right">(10)</div>

in this expression answers my daughter's question:

$$t_{\text{impact}} \approx 5,544,225 \text{ seconds} \approx 64.17 \text{ days} .$$
<div align="right">(11)</div>

- You might have little confidence in this result since I skipped a few steps. The integral was obtained by a computer program, and I flippantly stated that you can check the answer by differentiation. (I actually did[-1-] , but, again, I used Maple to carry out that differentiation. That's not as crazy as it sounds, the differentiation and integration algorithms in Maple are quite distinct.) However, to put your mind at rest, it turns out that the same number, a little more than 64 days, can be obtained numerically.

- The method I used is a simple modification of Euler's method. Let s_n be the approximation of the distance of $\mathcal{E}$ from $\mathcal{O}$ after n seconds, and let v_n be the approximation of the velocity of $\mathcal{E}$ after n seconds. Then the numerical

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[-1-] You might also try to obtain the formula (9) via Maple, including the calculation of the integration constant. It's not as straightforward as it may sound, you will find that Maple insists you are dividing by zero.

method is as follows:

$$s_0 = H, \quad v_0 = 0$$

$$\textbf{For } n = 0, 1, 2, \ldots, \quad \textbf{until } s_n < R \textbf{ Do}$$

$$v_{n+1} = v_n - \frac{GR^2}{s_n^2} h$$

$$s_{n+1} = s_n + v_{n+1} h$$

$$\tag{12}$$

- The parameter h in this outline is the standard notation for the time step, i.e., one second in the actual program. The method makes physical sense: we update the velocity, and then we use the new velocity to update the distance. The velocity is negative, and getting more so, since the positive direction is up, away from $\mathcal{O}$.

- The following very simple Fortran program actually runs the method:

```
1              implicit double precision (a-z)
2              Earth = 1.49E11
3              Sun = 0.696E9
4              Gravity = 274
5              time = 0
6              step = 1
7              distance = Earth
8              speed = 0
9     100      continue
10             time = time + step
11             speed = speed -
12        .           Gravity*Sun*Sun/distance/distance*step
13             distance = distance + speed*step
14             if (distance > Sun) go to 100
```

```
15          write(*,*) time, speed, distance
16          write(*,*) time/86400
17          stop
18          end
```

It's output is

```
1       5544225.0000000     -616199.28492109
695554688.88871
2       64.169270833333
```

The last number is the impact time measured in days, which equals the time measured in seconds and divided by 86,400, the number of seconds in one day. The numerical results are consistent with the analytical results.

• For the fun of it, and since we've spent all that work, let's compute how long it would take for the moon to fall onto the Earth. The radius of the Earth is 6,378,000 meters, gravity on the surface of the Earth is 9.8 meters per second squared, and the distance of the moon from the Earth is 384 million meters. Substituting these values into our formula gives

$$t_{\text{impact}} \approx 418,222 \text{ seconds} \approx 4.8 \text{ days} .$$
$$(13)$$

• Again, these numbers are consistent with the computational results.