More finite Elements

Line, interval: \( a = x_0 < x_1 < \ldots < x_n = b \)

piecewise quadratic \( C^1 \)

\[ z_i = \frac{1}{2} (x_{i-1} + x_i) \]

quadratic on each \( [x_{i-1}, x_i] \)

nodal values = function values at \( x_i, z_i \)

- goes with \( x_i \) 2 intervals
- goes with \( z_i \) 1 interval

The system now is no longer tridiagonal.
(exercise: think about its structure)

It's still \( C^0 \), but we now have a higher approximation order.
- piecewise cubic Hermite $C^1$

- Nodal Values, $u(x_i)$, $u'(x_i)$

$$u^b(x) = \sum_{i=0}^{n} u^b(x_i) h_i(x) + u^b(x) \bar{h}_i(x)$$

$$h_i(x_j) = \delta_{ij} \quad \bar{h}_i(x_j) = 0$$

Graphs

- exercise: compute error
Things get more complicated with 2 variables.

Rectangular grids

\[(x_i, y_j) \quad i = 0, \ldots, n_x \]
\[j = 0, \ldots, n_y\]

\[a_x = x_0 \leq x_1 \leq \ldots \leq x_{n_x} = b_x\]
\[a_y = y_0 \leq y_1 \leq \ldots \leq y_{n_y} = b_y\]

On a single rectangle use a function of the form

\[a + bx + cy + dxy = p(x, y)\]

\[(x_{i-1}, y_j) \quad (x_i, y_j)\]

\[(x_{i-1}, y_{j-1}) \quad (x_i, y_{j-1})\]
- This is called a bilinear function.
- Setting x or y to a constant value gives a function that is linear in the other variable.
- Get a function of the form

\[ f(x,y) = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} a_{ij} \psi_{ij}(x,y) \]

where \( \psi_{ij} \) is piecewise bilinear

\[ \psi_{ij}(x_i, y_j) = S_{ij} S_{j} \]

- The \( S_{ij} \) are nodal values.
- The support of \( \psi_{ij} \) is the set of four or fewer rectangles that share the point \((x_i, y_j)\).
- The function is globally continuous since along an edge shared by two rectangles the function on either side reduces to the same linear function.
It's easy to write down the bilinear interpolant on the unit square

\[
F(x, y) = F(0, 0) (1-x)(1-y) + F(1, 0) x(1-y) + F(0, 1) (1-x)y + F(1, 1) xy
\]

can get a similar function on a general rectangle by a linear change in \(x\) and a similar change in \(y\).

Exercise:

we could construct \(c\) elements on a rectangular grid by using bicubic functions, akin to cubic Hermite in one variable. Exercise!

Note: it's critical that the sides of the rectangle are aligned with the coordinate axes.
A rectangular grid is very convenient but very restrictive.

For example, local refinement of a grid, keeping the grid rectangular, is expensive.

Also: hard to adapt to non-rectangular geometries.
- more flexible are triangulations.

- A set $\Delta = \{ T_1, T_2, \ldots, T_N \}$ of triangles in the plane is called a triangulation of $\Omega = \bigcup T_i$ provided that if a pair of triangles in $\Delta$ intersect then their intersection is either a common vertex or a common edge.

- We look for nodal values at the vertices of the triangles.

- Given the vertices, the triangulation is not unique.

- This is in contrast with the partition of an interval.
- constructing triangulations is a big subject by itself.
- Books have been written on the subject, e.g.:

  Frank P. Preparata, M. I. Shamos
  (this is a graduate level monograph.)

- combinatorics of triangulations

  \[ N: \text{number of triangles} \]
  \[ V_B: \# \text{ of boundary vertices} \]
  \[ V_I: \# \text{ of interior vertices} \]
  \[ V = V_B + V_I \]
  \[ E_B: \# \text{ of boundary edges} \]
  \[ E_I: \# \text{ of interior edges} \]
  \[ E = E_B + E_I \]
- Triangulations in the plane are shellable, they can be built adding one triangle at a time

\[ \begin{array}{c}
\text{flap} \\
\text{fill}
\end{array} \]

- suppose we have \( V_B \) being vtes and \( V_I \) interior vtes.

Then

\[ E = 2V_B + 3V_I - 3 \]
\[ N = V_B + 2V_I - 2 \]

- proof by induction. true when \( V_B = 3 \) and \( V_I = 0 \)

- adding a flap adding a fill

\[ \begin{array}{c}
V_B \rightarrow V_B + 1 \\
V_I \rightarrow V_I \\
E \rightarrow E + 2
\end{array} \]
\[ \begin{array}{c}
V_B \rightarrow V_B - 1 \\
V_I \rightarrow V_I + 1 \\
E \rightarrow E + 1
\end{array} \]
\[ N \rightarrow N + 1 \]

- covariant heat functions.