Let's return to $u_t = u_{xx}$ and its semi-discretization.

\[ u_m(t) \approx u(x_m, t) \quad x_m = m h \]

\[ u_m' = \frac{1}{h^2} \left( u_{m+1} - 2u_m + u_{m-1} \right) \]

\[
\begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
u_1 \\ v_2 \\ v_3
\end{bmatrix} + \mathbf{v}(t)
\]

where $\mathbf{v}(t)$ is a vector that accounts for boundary conditions.

It's instructive to analyze the eigenvalues of

\[
A = \begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{bmatrix}
\]

The main reason for doing this now is...
- How do we find the eigenvalues?

Let's say one eigenvector is

\[ x = [x_1, \ldots, x_n] \]

\[ A \cdot x = \lambda x \] gives rise to the equations

\[ -2x_1 + x_2 = \lambda x_1 \]
\[ x_{k-1} - 2x_k + x_{k+1} = \lambda x_k \quad i = 2, \ldots, n-1 \]
\[ x_{n-1} - 2x_n = \lambda x_n \]

(2) can be rewritten as

\[ x_i - (2+\lambda)x_i + x_{i+1} = 0 \]

This is a homogeneous constant coefficient linear difference equation!

- We know how to solve it

\[ \phi(r) = r^2 - (2+\lambda)r + 1 = 0 \]

\[ \lambda = \frac{2+\lambda \pm \sqrt{(2+\lambda)^2 - 4}}{2} \]

\[ (2+\lambda)^2 - 4 = \lambda^2 + 4\lambda \]
\[ r = \frac{2 + \lambda \pm \sqrt{\lambda^2 + 4\lambda}}{2} \]

- Gersgorin's Theorem implies that
  \[-4 \leq \lambda \leq 0\]
- Thus \( \lambda^2 + 4\lambda \leq 0 \) and the roots of \( g \) are complex.
- Moreover,
  \[ |r| = \frac{1}{4} \left( (2+\lambda)^2 + 4 - (2+\lambda)^2 \right) = 1 \]
- We can therefore write
  \[ r = e^{\pm i\theta} \]
  for some \( \theta \).
- The general solution of (2) is
  \[ x_k = \alpha e^{i\theta} + \beta e^{-i\theta} \]  (4)
- The eigenvectors of \( A \) are real and so we must have \( \lambda = \bar{\beta} \)
- We get

\[ x_k = \lambda e^{ik\theta} + \bar{\lambda} e^{-ik\theta} = \]

\[ = \lambda (\cos k\theta + i\sin k\theta) + \bar{\lambda} (\cos k\theta - i\sin k\theta) \]

\[ = (\lambda + \bar{\lambda}) \cos k\theta + (\lambda - \bar{\lambda}) i\sin k\theta \]

\[ = \rho \cos k\theta + S \sin k\theta \]

where \( \rho \) and \( S \) are real.

- Eigen vectors are determined only up to a constant factor so we may assume that \( S = 1 \)

- We can accommodate the equations (1) and (3) by setting

\[ x_0 = x_{M+1} = 0 \]

- Thus \( x_0 = \rho e^{\cos \theta} + S \sin \theta = \rho^2 = 0 \)

and \( x_n = S \sin \theta \)

- \( x_{M+1} = 0 \) implies that \( \theta = \frac{j\pi}{M+1} \)

for some integer \( j \)
we get from

\[ e^{i\theta} = r = \frac{1}{2} (2 + z + \sqrt{z^2 + 4z}) \]

\[ = \cos \theta + isin \theta \]

that

\[ 1 + \frac{2}{z} = \cos \theta \]

or

\[ -z = 2(\cos \theta - 1) \]

\[ = 2(\cos \frac{j\pi}{M+1} - 1) \]

\[ j = 1, \ldots, M \]

- In particular, letting \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) be the eigenvalues with smallest and largest absolute value, respectively, we get

\[ \lambda_{\text{min}} = -2(1 - \cos \pi h) \quad h = \frac{1}{M+1} \]

\[ \lambda_{\text{max}} = -2(1 - \cos \frac{M\pi h}{M+1}) \]

- cool application of solving finite difference equations!
Recall that the eigenvalues of the Jacobian of an ODE correspond to the relevant time scales.

For our problem

\[ \lambda_{\max} = x - 4 \]
\[ \lambda_{\min} = -2(1 - \cos \pi h) \]
\[ x - 2(1 - (1 + 0h + \frac{(\pi h)^2}{2}) ) \]
\[ = (\pi h)^2 + O(h^3) \]

The ratio of the max and min time scale, therefore is

\[ \frac{4}{(\pi h)^2} \]

which means the system gets stiffer as \( h \) gets smaller.

What does that mean for \( \frac{1}{h^2} A \)?

This is reflected, for example, in the fact that for Euler's Method \( k = \frac{1}{h^2} \) goes to zero faster than \( h \).
- Classification of PDEs (example)

\[ a u_{xx} + b u_{xy} + c u_{yy} = f \]  \hspace{1cm} (*)

where, in general, \( a, b, c, f \) are functions of \( x, y, u, u_x, \) and \( u_y \)

- One of \( x \) or \( y \) might be time in this context.

- Now consider a curve

\[ \Gamma \]

\[ (x, y) = (x(t), y(t)) \]

- Suppose we know \( u, u_x, \) and \( u_y \) on \( \Gamma \). We ask under which condition does this information determine \( u_{xx}, u_{xy}, \) and \( u_{yy} \) on \( \Gamma \) such that (*) is satisfied.
- In what follows, evaluate \( u \) and its derivatives at \((x(t), y(t))\).

- If the derivatives exist we get

\[
\frac{d}{dt} u_x = u_{xx} \frac{dx}{dt} + u_{xy} \frac{dy}{dt} \\
\frac{d}{dt} u_y = u_{xy} \frac{dx}{dt} + u_{yy} \frac{dy}{dt}
\]

- Thus we obtain the linear system

\[
\begin{bmatrix}
\frac{dx}{dt} & \frac{dy}{dt} & 0 \\
0 & \frac{dx}{dt} & \frac{dy}{dt} \\
0 & 0 & \frac{dx}{dt}
\end{bmatrix}
\begin{bmatrix}
u_{xx} \\
u_{xy} \\
u_{yy}
\end{bmatrix}
= 
\begin{bmatrix}
f \\
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix}
\]

- The solution of this system exists and is unique unless the determinant of the coefficient matrix is zero, i.e.,

\[
a (\frac{dy}{dt})^2 - b \frac{dx}{dt} \frac{dy}{dt} + c (\frac{dx}{dt})^2 = 0
\]
Let's "multiply" with \((dt)^2\) and divide by \((dx)^2\). We get

\[
\alpha \left( \frac{dy}{dx} \right)^2 + b \frac{dy}{dx} + c = 0
\]

This is a quadratic equation for \(\frac{dy}{dx}\).

It has

- 2 real solutions if \(b^2 - 4ac > 0\) \(\text{hyperbolic}\)
- 1 real solution if \(b^2 - 4ac = 0\) \(\text{parabolic} (* )\)
- 0 real solutions if \(b^2 - 4ac < 0\) \(\text{elliptic}\)

We considered \(u_+ = u_{xx}\)

\[
\alpha = 1, \quad b = 0 = c, \quad b^2 - 4ac = 0
\]

parabolic!