Great Book on Finite Difference Methods for PDEs

John Strikwerda, "Finite Difference Schemes and Partial Differential Equations", SIAM publication, 2

I can't find my copy.

Let's talk about convergence.

As an example, let's go back to

\[ U_{m+1}^n = U_m^n + r (U_{m+1}^n - 2U_m^n + U_{m-1}^n) \]

\[ = (1-2r)U_m^n + r(U_{m+1}^n + U_{m-1}^n) \]

\[ u_t = u_{xx} \quad X,T \text{ points in the grid} \]

we want \[ \lim (u(X,T) - U(X,T)) = 0 \]

\[ X = x_m, T = t_n \]

\[ h, k \to 0 \]

\[ m, n \to \infty \]
- Let \( z_m^n = v(x_m, t^n) - U_m^n \) be the error at \((x, t)\).

- We know that

\[
U_m^{n+1} = (1-2v)U_m^n + v(U_{m+1}^n + U_{m-1}^n)
\]

and

\[
u(x_m, t^{n+1}) = (1-2v)u(x_m, t^n) + v(u(x_{m+1}, t^n) + u(x_{m-1}, t^n)) + O(k^2 + kh^2)
\]

- Thus

\[
z_m^{n+1} = (1-2v)z_m^n + v(z_{m+1}^n + z_{m-1}^n) + O(k^2 + kh^2)
\]

- Suppose \( 0 < v \leq \frac{1}{2} \).

- The coefficients on the right, \( v \) and \( 1-2v \), are both positive.

- Thus

\[
|z_m^{n+1}| \leq (1-2v)|z_m^n| + v|z_{m+1}^n| + v|z_{m-1}^n| + A(k^2 + kh^2)
\]

for some constant \( A \).
- Let $Z^{(n+1)} = \max_m |z^{(n)}_m|

we can take the maximum first on the right, then on the left

$$|z^{n+1}_m| \leq z^{(n)} + A(k^2 + kh^2)$$

- The right-hand side does not depend on $n$.

- Thus

$$Z^{(n+1)} \leq Z^{(n)} + A(k^2 + kh^2)$$

- $Z^{(0)} = 0$ (by the initial condition)

- By induction

$$Z^{(n)} \leq n A(k^2 + kh^2)$$

$$= nTA(k+h^2) \to 0$$

as $h,k \to 0$ $T$ fixed
Notes:

- In general it is hopeless to bound the GTE in terms of LTE. But this is a particularly simple case. A similar situation for ODEs would be to apply Euler's Method to \( y' = y \)

  (exercise)

- Note the significance of \( r \) and \( (1-2r) > 0 \). If \( r > \frac{1}{2} \) and then \( 1-2r < 0 \),

  we would get the recursion

  \[
  z^{(n+1)} \leq \left( |r| + |1 - 2r| \right) z^{(n)}
  \]

  and we would get exponential growth.

- we lost one power of \( k \) and none of \( h 

  Exercise: show that Crank Nicolson converges for all \( r \).
- crucial concept: domain of dependence of $(X, T)$

- It's the number of points that affect the value of $U_n$

\[ \Theta = \tan^{-1} \frac{h}{k} \]

- what is the associated domain of dependence for the DE?

- consider the Cauchy (piece initial value) problem

\[ u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \]

\[ u(x, 0) = g(x) \]

\[ u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp \left( \frac{-(x-z)^2}{4t} \right) g(z) \, dz \]

exercise
- It follows from this that the DoD of the PDE is infinite!

- One significance of \( r = \frac{k}{h^2} < \frac{1}{2} \)
\[ k < \frac{1}{2} h^2 \]

\[
\arctan \frac{h}{k} > \arctan \frac{h}{\frac{1}{2} h^2} = \arctan \frac{2}{h} \rightarrow \frac{\pi}{2}
\]

- In other words, the DoD of the numerical method approaches that of the PDE.

- Query: how can the DoD of the PDE be infinite? Didn't Einstein prove information can't travel faster than at the speed of light?

- What is the DoD for Crank-Nicolson?
\[(1+r)U_m^{n+1} - \frac{r}{2}(U_{m+1}^{n+1} + U_{m-1}^{n+1}) = (1-r)U_m^n + \frac{r}{2}(U_{m+1}^n + U_{m-1}^n)\]

Both methods we have discussed, Euler's and Crank Nicolson, for \(u_t = u_{xx}\) were derived by applying ODE methods (Euler and Trapezoidal) to the ODEs obtained by the method of lines.

Not all PDE methods can be so obtained.

Exercise consider the stencil and corresponding coefficients

\[
\begin{array}{ccc}
A & B & C \\
D & O & O \\
D & E & F
\end{array}
\]

giving the method

\[
AU_{m-1}^{n+1} + BU_m^{n+1} + CU_{m+1}^{n+1} = \\
= DU_{m-1}^n + EU_m^n + FU_{m+1}^{n+1}
\]
- and find coefficients $A_1, \ldots, F$ such that the LTE error order is as high as possible.

- $\text{LTE(CN)} = O(k^3 + kh^2)$

- The result is the Douglas Formula:

$$U_m^{n+1} = \frac{1}{2}(r-\frac{1}{c})(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}) =$$

$$= U_m^n + \frac{1}{2}(r+\frac{1}{c})(U_{m+1}^n - 2U_m^n + U_{m-1}^n)$$

- Exercises: $\text{LTE(DM)} = O(k^3 + kh^4)$

- verify the statement about the LTE

- Using the von Neumann approach show that the Douglas Formula is stable for all $r > 0$