Math 6620

This semester's subject:
Numerical solution of DEs

\[ \text{DE} \xrightarrow{\text{ordinary}} \text{IVP} \quad \text{BVP} \quad \text{ODE} \\
\quad \text{DE} \xleftarrow{\text{partial}} \text{IVP} \quad \text{BVP} \quad \text{PDE} \\
\quad \text{DE} \xleftarrow{\text{IVP/BVP}} \]

- not doing: Differential-Algebraic Systems
  \[ F(x, y, y') = 0 \]
  Integral Equations

- we will focus on a few major issues and illustrate those in as simple a context as possible
  - Discretization
  - Convergence \xleftarrow{\text{Consistency}} \text{Stability}
  - Error Propagation
Let's introduce some of the ideas in a simple context.

\[ y' = f(x, y), \quad a \leq x \leq b \quad y(a) = y_0 \]

Initial value problem of one ODE

The simplest numerical method which you may have first encountered as early as Calculus is Euler's Method

Leonard Euler (Oiler), 1707-1783, Swiss, most of his life he was almost blind on one eye, had 13 children (five survived childhood) and also published around 850 individual pieces of mathematical work. Most prolific mathematician ever, and one of the most influential. His work is now accessible online:

eulerarchive.maa.org

(most documents are in German or Latin, however)
- Discretize

\[ h: \text{ step-size} \quad h = \frac{b-a}{N} \]

\[ x_n = a + nh \quad n=0, 1, 2, \ldots \]

\[ y_n \approx y(x_n) \quad y_0 = y(a) \quad y_n = ? \quad n > 0 \]

\[ f_n = f(x_n, y_n) \]

\[ y = y(x) \]

\[ y_1 = y_0 + hf_0 \]

Idea: Follow the tangent

\[ \frac{y_1 - y_0}{h} = f(x_0, y_0) \]
however, \((x_0, y_0)\) is not on the (unknown) graph of the solution.

- Follow the solution of the IVP

\[ y' = f(x, y) \quad y(x_0) = y_0, \]

\[ y_2 = y_1 + hf_1, \]

- and so on.

\[ y_0 = y(x_0) \quad y_{n+1} = y_n + hf_n, \quad n = 0, 1, \ldots \]

**Euler's Method.**

- How does this work?

- It depends on \(h\), and the behavior of the neighboring solution.

- Following are a few worksheets.
- Consider the IVP

\[ y' = \cos x + 2(y - \sin x) \quad y_0 = \xi \]

- The solution, of course, is

\[ y(x) = \sin x + \xi e^{2x} \]

- Neighboring solutions
  - are parallel if \( \lambda = 0 \)
  - diverge if \( \lambda > 0 \)
  - converge if \( \lambda < 0 \)

- How does Euler's Method work?

- On the following pages, start at

\[ x_0 = 0 \quad y_0 = \alpha = 0 \]
\[ \lambda = 0 \]
\( \lambda = 1 \)
$z = -1$
$\lambda = 10$
$\lambda = -10$
Local and global truncation error.

"Truncation" refers to truncating a Taylor series

- we (usually) ignore round-off errors.

\[ GTE_{n+1} = Y(x_{n+1}) - Y_{n+1} \]

\[ LTE_{n+1} = Y(x_{n+1}) - Y(x_n) - h f(x_n, y(x_n)) \]

\[ = Y(x_{n+1}) - Y(x_n) - h y'(x_n) \]

(error under the "localizing assumption" that back information is exact.)

- we want to control the GTE but the LTE is easier to handle.
$$GTE_{n+1} = y(x_{n+1}) - y_{n+1}$$

$$= \frac{y(x_{n+1}) - y(x_n) - h y'(x_n)}{LTE_{n+1}}$$

$$+ y(x_n) + h f(x_n, y(x_n)) - y_n - h f(x_n, y_n)$$

$$= LTE_{n+1} + \frac{y(x_n) - y_n + h f_y(x,n)(y(x_n) - y_n)}{LTE_{n+1}}$$

$$= LTE_{n+1} + (1 + h f_y(x,n)) GTE_n$$

↑ ↑ ↑

new error amplification old error factor

- amplification factor has absolute value > 1 if \( f_y(x) > 0 \) or \( h f_y(x) < -1 \).

- Mathematical stability \( (f_y(x) < 0) \) does not imply numerical stability.