Hand in typeset answers to the following questions on

Friday, October 4,

immediately before the lecture. All questions have equal weight.


-1- (A triangular orthogonal matrix is diagonal.) Show that a triangular orthogonal matrix is diagonal. (GL, Problem 2.5.2, p. 74).

-2- (Small Pivots imply Ill-Conditioning.) Suppose

\[ P(A + E) = \hat{L}\hat{U} \quad (1) \]

where \( P \) is a permutation matrix, \( \hat{L} \) is lower triangular with

\[ |\hat{\ell}_{ij}| \leq 1, \quad (2) \]

and \( \hat{U} \) is upper triangular. Show that

\[ \|A\|_\infty \|A^{-1}\|_\infty \geq \frac{\|A\|_\infty}{\|E\|_\infty + \mu} \quad \text{where} \quad \mu = \min |\hat{u}_{ii}|. \quad (3) \]

(Thus, if a small pivot is encountered when Gaussian Elimination with pivoting is applied to \( A \), then \( A \) is ill-conditioned. The converse is not true.) Hints: This is an example of backward error analysis. I would first assume that \( E = 0 \). Note that picking \( E \neq 0 \) actually decreases the lower bound on the condition number, which provides a rare case where an error may make us hopeful to get better results than we deserve. Pivoting ensures that (2) is satisfied, but to simplify the notation a little you may assume that \( P = I \). (GL, problem 3.5.3, page 130.)

-3- (Inequalities are sharp.) Explain the meaning of

\[ \frac{1}{\|A\|\|A^{-1}\|} \leq \frac{\|r\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|r\|}{\|b\|} \]

and show how we derived these inequalities in class.

a. For a general matrix \( A \), and \( \| \cdot \| = \| \cdot \|_2 \), show that there are non-trivial examples (i.e., \( x \neq 0 \neq e \)) where the right hand inequality is satisfied with equality in (4). Do the same for the left hand inequality in (4).

b. Repeat part a. for \( \| \cdot \| = \| \cdot \|_\infty \)

-4- (Induced Matrix Norms.) Show that

\[ \|A\|_1 = \max_{j=1,...,n} \sum_{i=1}^{n} |a_{ij}|. \quad (5) \]
-5- **(Reverse Triangle Inequality.)** Show that for all vector norms \( \| \cdot \| \) and vectors \( x \) and \( y \):

\[
\| x - y \| \geq \| x \| - \| y \| \tag{6}
\]

and

\[
\| x + y \| \geq \| x \| - \| y \|. \tag{7}
\]

(see also Problem 2.2.6, page 54, in GL).

-6- **(Frobenius Norm.)** Show that there is no vector norm that induces the Frobenius Norm

\[
\| A \|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}. \tag{8}
\]

-7- **(Sherman-Morrison-Woodbury Formula.)** In the previous home work we saw that a rank one change of a matrix changes the inverse also by a rank one matrix. In this problem we generalize this result to rank \( k \geq 1 \). Suppose \( A \) is a non-singular \( n \times n \) matrix, and \( U \) and \( V \) are \( n \times k \) matrices such that \( A + UV^T \) and \( I + V^T A^{-1} U \) are non-singular. Show that

\[
(A + UV^T)^{-1} = A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}. \tag{9}
\]

(GL, page 50.)

-8- **(Singular Value Decomposition.)** Let \( w \in \mathbb{R}^n \) be a given vector. Interpret it as an \( n \times 1 \) matrix, and describe its Singular Value Decomposition. Do the same thing for \( w^T \).

-9- **(Pulling Rank.)** Show that if an \( m \times n \) matrix \( A \) has rank \( p \), then there exists an \( m \times p \) matrix \( X \) and an \( n \times p \) matrix \( Y \) such that \( A = XY^T \) where the ranks of \( X \) and \( Y \) are both \( p \). (GL, Problem 2.1.1, page 51.)

-10- **(More rank pulling.)** Show that if the \( m \times n \) matrix \( A \) has rank \( n \), then

\[
\| A(A^T)^{-1} A \|_2 = 1. \tag{10}
\]

Assume \( m \geq n \). Caution: It is not true that \( A(A^T)^{-1} A \) is diagonal! This innocuous looking problem will cause you to put many things together. You may find a different way of doing this, but I solved the problem by thinking about orthogonal projections and their singular values. (GL, problem 2.5.8, page 74.)

-11- **(Vector Norms.)** Show that if \( x \in \mathbb{R}^n \) then

\[
\lim_{p \to \infty} \| x \|_p = \| x \|_\infty. \tag{11}
\]

This is the reason for calling the infinity norm by that name. (GL, Problem 2.2.1, p. 54.)