Math 6610

- Let $f$ be $n+1$ times differentiable on $[a,b]$.

- Suppose we are given $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$.

- Let $p$ be the unique polynomial satisfying

  \[ p(x_i) = f(x_i) \quad i = 0, 1, \ldots, n \]

- Then for all $x$ in $(a,b)$ there exists $\xi \in (a,b)$ such that

  \[ f(x) - p(x) = \frac{(x-x_0)\ldots(x-x_n) f^{(n+1)}(\xi)}{(n+1)!} \]

- This is certainly true if $x = x_i$ for some $i$.
  (get $0 = 0$ regardless of $\xi$)

- So suppose $x \neq x_i; \quad i = 0, 1, \ldots, n$
Let
\[ F(t) = f(t) - p(t) - \frac{(t-x_0) \cdots (t-x_n)}{(x-x_0) \cdots (x-x_n)} (f(x) - p(x)) \]

Then \( F(t) = 0 \) for \( t = x_1, x_2, \ldots, x_n \)

Since \( F \) has \( n+2 \) distinct roots, by Rolle's generalized theorem
\[ F^{(n+1)}(\xi) = 0 \quad \text{for some } \xi \]

But
\[ F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{(n+1)!}{\prod_{i=0}^{n} (x-x_i)} (f(x) - p(x)) = 0 \]

Solving for \( f(x) - p(x) \) gives
the result
\[ f(x) - p(x) = \frac{\prod_{i=0}^{n} (x-x_i)}{(n+1)!} f^{(n+1)}(\xi) \]
This expression goes to zero as $n$ goes to zero if the derivatives of $f$ are bounded independently of $n$.

Examples:

$$f(x) = \sin x, \cos x, e^x$$

On the other hand, the derivatives may grow too fast, or may not exist.

Example. Runge Phenomenon.

$$f(x) = \frac{1}{1 + x^2}$$

$$[a, b] = [-5, 5] \quad x_i = -5 + \frac{10i}{n} \quad (\text{equally spaced points})$$
- you get oscillations towards the endpoints that grow arbitrarily large as \( n \) goes to infinity.

- On the other hand, the error goes to zero if the knots are the roots of the Chebychev polynomials

\[
x_i = 5 \cos \frac{i \pi}{n} \quad i = 0, \ldots, n
\]

knots are clustered towards the endpoints

- easy programming exercise.

- why do the derivatives of \( \frac{1}{1 + x^2} \) grow so fast? What's the trouble.
- Consider interpolation scheme

\[
\begin{align*}
    P_0 &:= x_0 \\
    P_1 &:= x_0, x_1 \\
    P_2 &:= x_0, x_1, x_2 \\
    \vdots \\
    P_n &:= x_0, x_1, \ldots, x_n
\end{align*}
\]

- For any interpolation scheme, you can find an arbitrarily often differentiable function \( f \) such that

\[
\max_{a \leq x \leq b} |f(x) - P_n(x)| \rightarrow \infty \quad \text{as } n \rightarrow \infty
\]

- For any continuous function \( f \), you can find an interpolation scheme such that

\[
\max_{a \leq x \leq b} |f(x) - P_n(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty
\]