Interpolation = Exact Representation of Data
- opposed to Approximation ... approximate representation of Data.

univariate: 1 independent variable
bi, tri, multi-variate
2, 3, or several independent variables.

Lagrange Interpolation: interpolate function values only.
several other types.

univariate Lagrange interpolation problem

Given data \((x_i, y_i)\) \(i = 0, \ldots, n\)
Find a (univariate) polynomial of degree \(n\) such that

\[ p(x_i) = y_i \quad i = 0, \ldots, n \]

\( y_i = f(x_i) \), perhaps
- The $x_i$ are called knots, nodes, abscissas, or data sites.
- The $y_i$ are "data" or "function values".
- Obviously, the data sites must be distinct:
  \[
  i \neq j \Rightarrow x_i \neq x_j
  \]
- This is necessary. It's also sufficient to have a unique solution
- Let $p(x) = \sum_{j=0}^{n} a_j x^j$
- we get a linear system

\[
V_n \alpha = y
\]

\[
V_n = \begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^n \\
1 & x_2 & x_2^2 & \cdots & x_2^n \\
1 & x_3 & x_3^2 & \cdots & x_3^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^n \\
\end{bmatrix}
\]

Vandermonde Matrix

\[
\alpha = [a_0, \ldots, a_n]^T
\]

\[
y = [y_0, \ldots, y_n]^T
\]
The Vandermonde Matrix is invertible provided the cluster sites are distinct.

- We'll see by induction that

\[ |V_n| = \prod_{j > i} (x_j - x_i) \]

\[ n = 1 \quad |V_1| = \begin{vmatrix} x_0 \\ x_1 \end{vmatrix} = x_1 - x_0 \]

- For the induction step set \( x_n = x \) and expand the determinant about the last row.

- Note that we get zero if the last row equals one of the other rows, i.e., \( x = x_i \), \( i = 0, \ldots, n-1 \).

- Thus \( |V_n| = \sum_{i=0}^{n} A_i x_i \)

\[ = A (x-x_0) (x-x_1) \ldots (x-x_{n-1}) \]

\[ = |V_{n-1}| (x-x_0)(x-x_1) \ldots (x-x_{n-1}) \]
Setting $x = x_n$, give

$$|V_n| = |V_{n-1}| (x_n - x_1) \cdots (x_n - x_{n-1})$$

$$= \prod_{j>i} (x_j - x_i) \neq 0 \quad \text{if } i \neq j \Rightarrow x_i \neq x_j$$

we can also construct the interpolating polynomial directly

$$p(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

$$L_i(x) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$L_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

This is the Lagrange form of the interpolating polynomial.

Also cardinal form

$L_i$: Lagrange or cardinal basis form.
- Lagrange form shows existence

- Since the system is square and linear
  \[ \text{Existence} \Rightarrow \text{Uniqueness} \]

- In both cases, if we want to add a new point, we'll essentially have to start over.

- In the Newton form we add one point at a time.

\[ P_k \text{ interpolates at } x_0, \ldots, x_k \]

\[ P_0(x) = y_0 \]

\[ P_1(x) = y_0 + p_0(x-x_0) \]

\[ P_i(x) = y_0 + p_1(x-x_0) - \delta_i = \frac{y_i - y_0}{x_i - x_0} \]

**Divided Difference**
In general
\[ P_{k+1}(x) = P_k(x) + \frac{k}{k+1} \prod_{i=0}^{k} (x-x_i) \]

\[ P_{k+1}(x_{k+1}) = y_{k+1} = P_k(x_{k+1}) + \frac{k}{k+1} \prod_{i=0}^{k} (x_{k+1} - x_i) \]

\[ f_{k+1} = \frac{y_{k+1} - P_k(x_{k+1})}{\prod_{i=0}^{k} (x_{k+1} - x_i)} \]

Blended or Iterated Interpolation

\( P_0 \) interpolates at \((x_0, y_0), \ldots, (x_k, y_k)\)

\( P_1 \) interpolates at \((x_1, y_1), \ldots, (x_{k+1}, y_{k+1})\)

\( P \) interpolates at \((x_0, y_0), \ldots, (x_{k+1}, y_{k+1})\)

\[ P(x) = \frac{x-x_{k+1}}{x_0-x_{k+1}} P_0(x) + \frac{x-x_0}{x_{k+1}-x_0} P_1(x) \]

General idea: blend two particular interpolants to get an interpolant that combines the properties of both.
\[ p(x_0) = P_0(x_2) = y_0 \]
\[ p(x_{k+1}) = P_1(x_{k+1}) = y_{k+1} \]
\[ 0 < i < k + 1 \]
\[ p(x_i) = \left( \frac{x - x_{k+1}}{x_0 - x_{k+1}} + \frac{x - x_0}{x_0 - x_{k+1}} \right) y_i = y_i \]

\[
\begin{align*}
& x_0, y_0 \\
& x_1, y_1 \\
& x_2, y_2 \\
& \vdots \\
& x_{n-1}, y_{n-1} \\
& x_n, y_n
\end{align*}
\]

- we discussed these various techniques to illustrate several ideas that occur in other contexts:
  - Linear system, standard form
  - Cardinal form
  - Adding one point at a time
  - Blending

Vandermonde
- However, all of these approaches get the same interpolant, just in a different form.

- Yet one more proof of uniqueness: the homogeneous problem $y_i = 0$, $i = 0, \ldots, n$, has a unique solution, $p(x) = 0$, exact because a polynomial of degree $n$ has only $n$ roots.

- Uniqueness of homogeneous version implies existence of general version.

- Existence and Uniqueness are not trivial.

- Example 1. $p(x) = ax^2 + bx + c$

  $p(-1) = A$  $p(1) = B$  $p(2) = C$

  $A = B = 0 \Rightarrow C = 0$
- Linear interpolation in two variables
  \[ L(x, y) = Ax + By + C \]
  \[ L(x_i, y_i) = z_i \quad i = 1, 2, 3 \]

- The three points \((x_i, y_i)\) must not lie on the same line

- Note: Error Analysis.