Math 6610

- Eigenvalue Problem: A \( n \times n \) throughout

\[ A\mathbf{x} = \lambda \mathbf{x} \quad \mathbf{x} \neq \mathbf{0} \]

\( \lambda \) eigenvalue
\( \mathbf{x} \) corresponding eigenvector

eigenvectors are determined only up to a (non-zero) factor

- \( A\mathbf{x} = \lambda \mathbf{x} \Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0} \Rightarrow \det(A - \lambda I) = 0 \)

characteristic polynomial

- A matrix also has left eigenvectors

\[ \mathbf{y}^T A = \lambda \mathbf{y}^T \]

- same eigenvalues.

- Left and right eigenvectors corresponding to distinct eigenvalues are biorthogonal.
\[ A x = \lambda x \quad \lambda \neq \mu \quad y^T A = \mu y^T \]
\[ y^T A x = 2 y^T x \quad y^T A x = \mu y^T x \]
\[ (\lambda - \mu) y^T x = 0 \Rightarrow y^T x = 0 \]

- most useful: Assume \( A \) has \( n \) linearly independent eigenvectors and just one eigenvector for \( \lambda \).

\[ A x^{(i)} = \lambda_i x^{(i)} \quad i = 1, \ldots, n \]
\[ \lambda_i \neq \lambda_j \quad i = 2, \ldots, n \]
\[ x = \sum_{i=1}^{n} \lambda_i x^{(i)} \]

- left eigenvectors

\[ y^{(i)} A = \lambda_i y^{(i)} \]
\[ y^{(i)} T x = \sum_{i=1}^{n} \lambda_i y^{(i)} T x^{(i)} \]
\[ = \lambda_i y^{(i)} T x^{(i)} \]
\[ \lambda_i = \frac{y^{(i)} T x}{y^{(i)} T x^{(i)}} \]
However, a matrix may not have any linearly independent eigenvectors. If it does not, it is called defective.

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{eigenvalues are } 0, 0 \]

\[ \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ y = 0 \]

only eigenvector is \[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

But suppose \( A \) is non-defective.

\[ x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(n)} \end{bmatrix} \quad \text{invertible} \]

\[ \Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \]
Then

\[ AX = X \Lambda \]
\[ \Lambda = X^{-1}AX \]

- This is an example of a similarity transform

\[ B = S^{-1}AS \]

- Similarity transforms preserve eigenvalues

\[ Ax = \lambda x \implies BS^{-1}x = S^{-1}ASS^{-1}x = \lambda S^{-1}x \]

- Of course it would be particularly nice if we had an orthogonal similarity transform

- Unitary for complex matrices

\[ Q^H = \overline{Q}^T = Q^{-1} \]

- Schur decomposition \( A \in \mathbb{C}^{n \times n} \)
There exists a unitary matrix $Q$ such that

$$Q^H A Q = D + N$$

where $D$ is diagonal and $N$ is strictly upper triangular ($i > j \Rightarrow N_{ij} = 0$).

- We can make $N$ as sparse as possible.

- This gives rise to the Jordan canonical form:

$$x^{-1} A x = \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \vdots \\ 0 & & \gamma_n \end{bmatrix} = D$$

- $J_i = \begin{bmatrix} \lambda_i & 1 \\ & \ddots & \ddots \\ & & \lambda_i \end{bmatrix}$ a "Jordan block".

- $J_i$ has only one eigenvector!
- **Algebraic multiplicity** of an eigenvalue: 
  \# of times it appears on the diagonal of $J$

- **Geometric multiplicity**: the number of Jordan blocks that have that eigenvalue on the diagonal.

- **Example**

\[
J = \begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 1 \\
0 & 0 & 3
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>alg. mult</th>
<th>geometric mult</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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</tr>
</tbody>
</table>
- **Backward Error Analysis**

Recall linear system

\[ A(x+e) = b + r \]

\[
\frac{\|e\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|r\|}{\|b\|}
\]

- Can we get something similar for the eigenvalue problem?

- Yes, but use different approach

\[
(A + \varepsilon F)x(\varepsilon) = 2x(\varepsilon)x(\varepsilon)
\]

\( \|F\| = 1 \)

\( \|x\| = 1 \)

\( \|y\| = 1 \)

\( \|y^T x\| = 1 \)

Also:

\[ A x = 2 x \]

\[ y^T A = 2 y^T \]

\[ y^T x = 1 \]

- Differentiate in (*) with respect to \( \varepsilon \)

\[
F x(\varepsilon + \varepsilon F) x(\varepsilon) = 2 x(\varepsilon)x(\varepsilon) + 2 x(\varepsilon)x(\varepsilon)
\]

Set \( \varepsilon = 0 \)

\[ x(0) = x \]

\[ \lambda(0) = 2 \]

\[ F x + A x = 2 x + 2 x \]
- left multiply with $y^T$

$$y^TFx + y^T A x = 2 y^T x + 2 y^T \bar{x}$$

$$2 y^T x = \bar{x}$$

$$y^TFx = \bar{x}$$

$$|2x| \leq ||y^T|| \| F \| \| x \| = ||y^T||$$

- so the greater $||y^T||$ the more sensitive is $\bar{x}$.

- $1 = 1 y^T x 1 = ||y^T|| \| x \| \cos \alpha = ||y^T x \|$$

$$\alpha: \text{angle formed } y \times \text{and } y$$

$$||y^T|| = \frac{1}{\cos \alpha}$$

- so the larger the angle between a right and corresponding left eigenvector, the more sensitive the eigenvalue.
Examples:

1. A symmetric \( x = y \), \( z = 0 \) as good as it gets!

2. \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ x^T y = 0 \]

as bad as it gets

Now think in terms of all eigenvectors
\[ A x^{(i)} = \lambda_i x^{(i)} \quad i = 1, \ldots, n \]

\[ X = \begin{bmatrix} x^{(1)} & \cdots & x^{(n)} \end{bmatrix} \]

\[ \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \]

\[ A X = X \Lambda \]

\[ x^T A x = \Lambda \]
The left eigenvectors are the rows of $X^{-1}$.

So the conditioning of the eigenvalue problem is given by

$$\|X^{-1}\| \|X\|$$

corresponds to $\|A\| \|A^{-1}\|^{-1}$ for linear systems.

defective $\leftrightarrow$ singular

Conditioning of linear system and eigenvalue problems are unrelated.