Math 6610

- Operation counts

\[ LU: \quad \text{For } k = 1, \ldots, n-1 \]
\[ \text{For } i = k+1, \ldots, n \]
\[ a_{ik} = \frac{a_{ik}}{a_{kk}} \]
\[ \text{For } j = k+1, \ldots, n \]
\[ a_{ij} = a_{ij} - a_{ik} a_{kj} \]

\[ \# \text{ Flops} = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} (1 + \sum_{j=k+1}^{n} 1) = \frac{n^3}{3} - \frac{n}{2} + \frac{n}{12} \]
\[ = \frac{n^3}{3} + O(n) \]

- 1 Flop = 1 mult or div

- we can simplify the nested sums using the well-known formulas for \( \sum_i \) and \( \sum i^2 \), or using maple, e.g.

\[ \text{simplify} \left( \text{sum}(\text{sum}(1 + \text{sum}(1, j=k+1..n), i=k+1..n), k=1..n) \right) \]

- However, usually only the leading term matters and we can get it quickly by integration
\[
\text{Flops} \times \sum_{i=0}^{n-1} \int_{k+1}^{n} (1 + \sin d_j) \, dk \, dk' = \frac{n^3}{3} - \frac{3}{2} n^2 + 2n - \frac{5}{6}
\]

\[= \frac{n^3}{3} + O(n^2)\]

- Here's another view

\[
\int_{c}^{Z} z^2 \, dz = \frac{n^3}{3}
\]

- Let's revisit matrix multiplication.

\[C = AB, \quad A, B, C \text{ } n \times n\]

\[c_{ij} = \sum a_{ik} b_{kj}, \quad n \cdot n^2 = n^3 \text{ multi}\]

- Can we do better?
\[
A = \begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}
\quad
B = \begin{bmatrix}
  B_{11} & B_{12} \\
  B_{21} & B_{22}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
  A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\
  A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22}
\end{bmatrix}
\]

- Multiplication of block matrices works just like ordinary matrix multiplication provided the dimensions match up.
Strassen Multiplication

Suppose \( m_1 = m_2 = n_1 = n_2 = \frac{n}{2} \)

Then \( C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \)

can be computed as follows:

\[
\begin{align*}
P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})B_{11} \\
P_3 &= A_{11}(B_{12} - B_{22}) \\
P_4 &= A_{22}(B_{21} - B_{11}) \\
P_5 &= (A_{11} + A_{12})B_{22} \\
P_6 &= (A_{21} - A_{11})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22}) \\
C_{11} &= P_1 + P_4 - P_5 + P_3 \\
C_{12} &= P_3 + P_5 \\
C_{21} &= P_2 + P_4 \\
C_{22} &= P_1 + P_3 - P_2 + P_6
\end{align*}
\]
check by substitution, for example

\[ C_{11} = P_1 + P_4 - P_5 + P_7 \]
\[ = (A^{11} + A^{22})(B^{11} + B^{22}) \]
\[ + A^{22}(B_2 - B_1) \]
\[ - (A^{11} + A^{12})B_2 \]
\[ + (A^{12} - A^{22})(B_2 + B_2) \]

\[ = A^{11}B_1 + A^{11}B_2 + A^{22}B_1 + A^{22}B_2 + A^{22}B_2 - A^{11}B_2 \]
\[ - A^{11}B_2 - A^{22}B_1 - A^{22}B_2 - A^{22}B_2 + A^{12}B_2 \]

\[ + A^{12}B_2 \]

\[ = A^{11}B_1 + A^{12}B_2 \]

- we only use 7 (instead of \( \leq 3 \))

- multiplications are \( O(n^2) \) instead of \( O(n^3) \)

- Matrix additions are \( O(n^2) \) instead of \( O(n^3) \)
Instead of $n^3$ operations we use

$$7 \cdot \left(\frac{n}{2}\right)^3 = \frac{7}{8} n^3 \text{ flops}$$

- Now, we can recur on this idea!
- suppose $n = 2$
- The overall effort is

$$\left(\frac{7}{8}\right)^5 n^3 \text{ flops}$$

- $$\left(\frac{7}{8}\right)^5 n^3 = 7 \cdot \frac{5}{2^3} = 7 \cdot \frac{\ln_2 n \cdot 2^3}{2^3}$$

$$= \left(\frac{7 \cdot 2^3}{2^3}\right) \ln_2 n = 7 \cdot \ln_2 n$$

For small matrices the number of additions and subtractions is significant, but we don't have to recur to the $1 \times 1$ level.