$C^1$ elements on triangles

- Recall: we need to work on only one element (triangle).
- We generate a general element matrix and then assemble the global linear system.
\( Q_{21} \) A **piecewise quintic \( C^1 \) element**

- Quintic have 21 parameters.
- The nodal data are: Function, gradient, and Hessian at the vertices.
- Values of the perpendicular Cross-Boundary Derivatives at the midpoints of edges.

\[
p(x, y) = \sum_{i+j \leq d} a_{ij} x^i y^j
\]

\[
\dim P_d = \binom{d+2}{2}
\]

- \( f(x, y) \)
- \( \nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \)
- \( \nabla^2 = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \)
$Q_{18}$ Condensation of Parameters

- we prefer to have nodal values only at vertices. The user may not even know about the triangulation!

- Start with $Q_{21}$ and require that the perpendicular cross-boundary derivatives be cubic.

$\text{perp. cross-boundary derivative along the edge is } \text{cubic} \quad \text{(instead of quartic)}$
• A common requirement is to have as few parameters as possible subject to requirements such as global smoothness or reproduction of polynomials of degree as high as possible.

• \( Q_{21} \) has 21 parameters per triangle, or 6 per vertex and 1 per edge.

• \( Q_{18} \) has 18 per triangle, and none per edge, or 6 per vertex globally.
Clough-Tocher

- Divide each (macro) triangle into 3 micro-triangles about the centroid.
- Use piecewise cubic $C^1$ functions on each macro triangle.
- Nodal data: function and gradient values at each vertex, perpendicular cross-boundary derivatives on midpoints of edges (of the macro triangle).

\[ \dim S_3^1 = 3 \cdot 10 - 3 \cdot 4 - 3 \cdot 3 + 3 = 12 \]
Condensation of parameters for Clough-Tocher

\[ Pcb \text{ along edges of macro-} \Delta \]
\[ \text{are linear instead of quadratic} \]
Summary

- The following Table lists
  - The name of the scheme
  - The polynomial degree, $d$
  - The degree of global smoothness, $r$
  - The nodal data. $V$ means vertex, $E$ means edge, $C$ means centroid. $f$ means function values, $\nabla f$ gradients, $\nabla^2 f$ Hessians, CB perpendicular cross-boundary derivatives
  - The number of data per triangle, $D$
  - The polynomial precision, $p$

<table>
<thead>
<tr>
<th>name</th>
<th>$d$</th>
<th>$r$</th>
<th>data</th>
<th>$D$</th>
<th>$p$</th>
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<td>1</td>
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<td>$C^0$</td>
<td>$f(V), f(E)$</td>
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<td>3</td>
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<td>5</td>
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<td>3</td>
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<tr>
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<td>$C^1$</td>
<td>$f(V), \nabla f(V)$,</td>
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