Basic Methods

for our standard IVP

\[ y' = f(x, y), \quad y(a) = y_0 \]  

1. We assume for the time being that \( y \) and \( f \) are scalar (i.e., \( m = 1 \)) and there is a unique solution of \( y' = f(x, y) \) through every point \((x_0, y_0)\) in the plane.

2. For the graphs in these notes let

\[ y' = -\sin x + \lambda(y - \cos x), \quad y(0) = 1 \]  

for specific values of \( \lambda \).

3. It’s easy check that the solution of the IVP (2) is

\[ y(x) = \cos x \]

and the general solution of the differential equation is

\[ y(x) = \cos x + C e^{\lambda x} \]

where

\[ C = y(0) - 1 \]

can determined by the initial value of \( y \).

4. The following graphs show some solutions for several values of \( \lambda \)
Figure 1. $\lambda = 0$.

- $\lambda = 0$ means that neighboring solutions are “parallel”, the slope is independent of $y$ (as in the case of quadrature).
Euler's Method

\[ y' = f(x, y) \]

\[ h f(x_0, y_0) = y_1 - y_0 \]

\[ \begin{align*}
  x_0 &= a \\
  x_1 &= x_0 + h \\
  x_2 &= x_1 + h \\
  &
\end{align*} \]

\[ \begin{align*}
  y_1 &= y_0 + h f(x_0, y_0) \\
  y_2 &= y_1 + h f(x_1, y_1) \\
  &
\end{align*} \]

E. M. \[ y_0 \text{ given} \]

\[ y_{n+1} = y_n + h f(x_n, y_n) \]

\[ n = 0, 1, 2, \ldots \]
Discretization

- The DE is infinite dimensional, computers are finite.
- Move through the interval in steps.
- Let \( x_n = a + nh \) where we assume for the moment that the step-size or discretization parameter \( h \) is constant.
- Then, for \( n = 0, 1, 2, \ldots \) let \( y_n \approx y(x_n) \). Our central problem is how to compute \( y_n \).
- If we follow the tangent we get Euler’s Method:

\[
y_{n+1} = y_n + hf(x_n, y_n)
\]

\( f_n = f(x_n, y_n) \) explicit
• The following Figures show what can happen with Euler’s Method.

**Figure 2.** \( \lambda = 1 \).
Figure 3. $\lambda = -1$. 
Figure 4. $\lambda = 10$. 
Figure 5. $\lambda = -10$. 
Figure 6. $\lambda = -100$. 
The Backward Euler Method

\[ y_{n+1} = y_n + h f(x_{n+1}, y_{n+1}). \]
The Trapezoidal Rule

\[ y_{n+1} - y_n = \frac{h}{2} \left( f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right). \]
Simpson’s Rule

\[ y_{n+1} - y_n = \frac{h}{3} (f(x_n, y_n) + 4f(x_{n+1}, y_{n+1}) + f(x_{n+2}, y_{n+2})) \]

starting problem

\[ y_1 = ? \]
Summary of Main Ideas

• **Discretization.** Computers can solve only finite dimensional problems.

• Step through the interval.

• new errors are introduced at each step. (*Local Accuracy*)

• But errors also impact future errors. (*Global Accuracy, Stability, Error Propagation*)

• The approximation at the new step can be given explicitly or implicitly as the solution of a system of $m$ equations.

• The approximation at the step may depend on the approximation of the just the last step (*one-step method*) or the approximations at several previous steps (*multistep methods*).

• For multistep methods we also have a *starting problem.*
Adams Methods

Adams - Bashforth, explicit
\[ y_{n+k} - y_{n+k-1} = h \sum_{j=0}^{k-1} \beta_j f(x_{n+j}, y_{n+j}) \]

Adams - Moulton, implicit
\[ y_{n+k} - y_{n+k-1} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}) \]

Backward Differentiation
\[ \sum_{j=0}^{k} \beta_j y_{n+j} = h \beta_k f(x_{n+k}, y_{n+k}) \]

starting?
Runge-Kutta Methods

\[ y_{n+1} = y_n + h \phi(x_n, y_n, h) \]

\[ \phi(x, y, h) = \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) \]

\[ k_1 = f(x, y) \]
\[ k_2 = f\left(x + \frac{h}{2}, y + \frac{h}{2} k_1 \right) \]
\[ k_3 = f\left(x + \frac{h}{2}, y + \frac{h}{2} k_2 \right) \]
\[ k_4 = f(x + h, y + h k_3) \]