Basic Methods

for our standard IVP

\[ y' = f(x, y), \quad y(a) = y_0 \] (1)

- We assume for the time being that \( y \) and \( f \) are scalar (i.e., \( m = 1 \)) and there is a unique solution of \( y' = f(x, y) \) through every point \((x_0, y_0)\) in the plane.

- For the graphs in these notes let

\[ y' = -\sin x + \lambda(y - \cos x), \quad y(0) = 1 \] (2)

for specific values of \( \lambda \).

- It’s easy check that the solution of the IVP (2) is

\[ y(x) = \cos x \]

and the general solution of the differential equation is

\[ y(x) = \cos x + Ce^{\lambda x} \]

where

\[ C = y(0) - 1 \]

can determined by the initial value of \( y \).

- The following graphs show some solutions for several values of \( \lambda \)
- $\lambda = 0$ means that neighboring solutions are “parallel”, the slope is independent of $y$ (as in the case of quadrature).
Euler’s Method
Discretization

- The DE is infinite dimensional, computers are finite.
- Move through the interval in steps.
- Let \( x_n = a + nh \) where we assume for the moment that the step-size or discretization parameter \( h \) is constant.
- Then, for \( n = 0, 1, 2, \ldots \) let \( y_n \approx y(x_n) \). Our central problem is how to compute \( y_n \).
- If we follow the tangent we get Euler’s Method:

\[
y_{n+1} = y_n + h f(x_n, y_n)
\]
• The following Figures show what can happen with Euler’s Method.

Figure 2. $\lambda = 1$. 

Figure 3. $\lambda = -1$. 
Figure 4. $\lambda = 10$. 
Figure 5. $\lambda = -10$. 
Figure 6. $\lambda = -100$. 
The Backward Euler Method

\[ y_{n+1} = y_n + hf(x_{n+1}). \]
The Trapezoidal Rule

\[ y_{n+1} - y_n = \frac{h}{2} \left( f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right). \]
Simpson’s Rule

\[ y_{n+1} - y_n = \frac{h}{3} \left( f(x_n, y_n) + 4f(x_{n+1}, y_{n+1}) + f(x_{n+2}, y_{n+2}) \right) \]
Summary of Main Ideas

- **Discretization.** Computers can solve only finite dimensional problems.

- Step through the interval.

- new errors are introduced at each step. (Local Accuracy)

- But errors also impact future errors. (Global Accuracy, Stability, Error Propagation)

- The approximation at the new step can be given explicitly or implicitly as the solution of a system of $m$ equations.

- The approximation at the step may depend on the approximation of the just the last step (one-step method) or the approximations at several previous steps (multistep methods).

- For multistep methods we also have a starting problem.
Adams Methods
Runge-Kutta Methods