\section*{$C^1$ Elements on Triangles}

- For tensor product grids we matched partial derivatives. That works well when grid lines parallel the coordinate axes.
- Triangles are more complicated.
- The key to dealing with them is the concept of \textbf{directional derivatives}/
- Suppose $e$ and $x$ are vectors in $\mathbb{R}^2$. (This actually works in $\mathbb{R}^b$).
- Then we define the \textbf{directional derivative of $f$ in the direction of $e$} as

$$D_e f = \frac{\partial}{\partial e} f(x) = \frac{d}{dt} f(x + te) \bigg|_{t=0}.$$ 

\begin{itemize}
  \item In Calculus $e$ is usually required to be a unit vector.
  \item However, we don’t need the assumption that $e$ is a unit vector.
  \item For example, $e$ might be an edge of a triangle.
\end{itemize}
• If $e = 0$ then

$$\frac{\partial}{\partial e} f(x) = 0$$

since $f(x + te)$ is constant.

• If $e = i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $D_e = \frac{\partial}{\partial x}$ and if $e = j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $D_e = \frac{\partial}{\partial y}$.

• Moreover, the gradient is given by

$$\nabla f = \begin{bmatrix} D_i f \\ D_j f \end{bmatrix}$$

and

$$D_e f = \nabla f \cdot e. \quad (1)$$

• If we pick two linearly independent directions $e_1$ and $e_2$ we can compute the gradient from $D_{e_1} f$ and $D_{e_2} f$ using (1).

Thus any two first order directional derivatives determine the gradient.

• Similarly, for second derivatives we get

$$D_{e_1} D_{e_2} f = e_1^T \nabla^2 f e_3$$

where

$$\nabla^2 f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$
So the two first and second order derivatives in two linearly independent direction determine all first and second order partial derivatives.

The significance of all this is that to get differentiability along an edge we have to consider continuity only of one derivative in a direction across the edge since the continuity of the derivative in the direction of the edge follows from continuity.
• Can we use piecewise cubics?
• Quartics?
• Quintics?
• Clough-Tocher