Assembly of Linear System

- Basic Theme: we assemble the linear system (the “stiffness equations”) by doing one subinterval at a time, instead of row by row.

- Return to our VP

\[ \int_{0}^{\pi} u'^2 + u^2 - 2uf \, dx = \text{min} \]

with piecewise linear elements

\[ \phi_j(x) = \begin{cases} 
\frac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j] \\
\frac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}] \\
0 & \text{else}
\end{cases} \]

where

\[ x_n = nh \quad \text{and} \quad h = \frac{\pi}{N}. \]

- Writing our approximant as

\[ u^h(x) = \sum_{i=0}^{N} \alpha_i \phi_i(x) \]

we have to solve the linear system (also called the global stiffness system)

\[ KU = F \]
where

\[ K_{ij} = \int_0^\pi \phi'_i \phi'_j + \phi_i \phi_j \, dx, \]
\[ U_j = \alpha_j \]
and
\[ F_j = \int_0^\pi \phi_k f \, dx \]

and \( k = 0, 1, \ldots N. \)

- In our particular case we obtained the linear system

\[ KU = F \]

where \( U \) is the vector of coefficients \( q_j \), the entries of \( F \) are \( \int_0^\pi \phi_j f \), and the coefficient matrix \( K \) is

\[ K = K_0 + K_1 \]

where

\[
K_1 = \frac{h}{6} \begin{bmatrix}
2 & 1 & 0 & \ldots & 0 & 0 & 0 \\
1 & 4 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & \vdots & \ddots & \ddots & \vdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 4 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1 & 2
\end{bmatrix}
\]
\[ K_0 = \frac{1}{h} \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1 & 2 & -1 \\
0 & 0 & 0 & \ldots & 0 & -1 & 1
\end{bmatrix} \]

We may have to modify the first and last equation depending on the boundary conditions. Think about precisely what kind of boundary conditions apply to the system as given above.

- Previously we computed the equations in (1) one at a time.

- However, it is more common to **assemble** the equations one subinterval at a time.

- This will become particularly convenient in the PDE case.

- This gives rise to an **element matrix** and the global matrix \( K \) is built from the element matrices.

- Basically we set up the system on a single subinterval and then assemble the global system from the system on the subinterval.
• The starting point is that for any function $g$
\[
\int_0^\pi g(x)dx = \sum_{j=1}^N \int_{x_{j-1}}^{x_j} g(x)dx.
\]

• So let’s consider
\[
I_j = \int_{x_{j-1}}^{x_j} \left( \sum_{k=0}^N \alpha_k \phi_k'(x) \right)^2 + \left( \sum_{k=0}^N \alpha_k \phi_k(x) \right)^2 \, dx
\]
\[
= \int_{x_{j-1}}^{x_j} \left( \alpha_{j-1} \phi_{j-1}'(x) + \alpha_j \phi_j'(x) \right)^2 + \left( \alpha_{j-1} \phi_{j-1}(x) + \alpha_j \phi_j(x) \right)^2
\]

• Clearly
\[
I = \int_0^\pi \left( u^h'(x) \right)^2 + \left( u^h(x) \right)^2 \, dx = \sum_{j=1}^N I_j.
\]

• For the purposes of our discussion, we ignore the $f$ terms, they only contribute to the right hand side.

• Notice that only the terms with $k = j$ and $k = j - 1$ contribute to the sum in $I_j$.

• Consider, for example,
\[
\frac{\partial}{\partial \alpha_j} \left( \alpha_{j-1} \phi_{j-1} + \alpha_j \phi_j \right)^2 = 2(\alpha_{j-1} \phi_{j-1} + \alpha_j \phi_j) \phi_j.
\]
and

\frac{\partial}{\partial \alpha_{j-1}} (\alpha_{j-1} \phi_{j-1} + \alpha_j \phi_j)^2 = 2(\alpha_{j-1} \phi_{j-1} + \alpha_j \phi_j) \phi_{j-1}

- The term involving the derivatives of the \( \phi \)'s works similarly. Putting things together we get

\frac{\partial}{\partial \alpha_j} I_j = 2 \int_{x_{j-1}}^{x_j} \left( \phi_{j-1} \phi_j + \phi_{j-1}' \phi_j' \right) \alpha_{j-1} + \left( \phi_j^2 + \phi_j'^2 \right) \alpha_j \, dx

and, similarly,

\frac{\partial}{\partial \alpha_{j-1}} I_{j-1} = 2 \int_{x_{j-1}}^{x_j} \left( \phi_{j-1}^2 + \phi_{j-1}'^2 \right) \alpha_{j-1} \phi_j \, dx + \left( \phi_j \phi_{j-1} + \phi_j' \phi_{j-1}' \right) \alpha_j \, dx

- Moreover, we get for the individual integrals:

\int_{x_{j-1}}^{x_j} \phi_{j-1}'^2 (x) \, dx = \int_{x_{j-1}}^{x_j} \phi_j'^2 (x) \, dx = \int_{x_{j-1}}^{x_j} \frac{1}{h^2} \, dx = \frac{1}{h}

\int_{x_{j-1}}^{x_j} \phi_{j-1}' (x) \phi_j' (x) \, dx = -\frac{1}{h}

\int_{x_{j-1}}^{x_j} \phi_{j-1}^2 (x) \, dx = \int_{x_{j-1}}^{x_j} \phi_j^2 (x) \, dx = \int_0^h \left( \frac{x}{h} \right)^2 \, dx = \frac{h}{3}

\int_{x_{j-1}}^{x_j} \phi_{j-1} (x) \phi_j (x) \, dx = \int_0^h \frac{x}{h} \frac{h-x}{h} = \frac{h}{6}
• Ignoring the factor 2 on both sides we get the element version of our global linear system:

$$\begin{bmatrix}
\frac{\partial}{\partial \alpha_{j-1}} I_j \\
\frac{\partial}{\partial \alpha_j} I_j
\end{bmatrix} = \left( \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} \alpha_{j-1} \\ \alpha_j \end{bmatrix}$$

• Strang refers to the first matrix on the right as the element stiffness matrix and the second as the element mass matrix.

• To obtain the global linear system we assemble—the global stiffness matrix $K_1$ and the global mass matrix $K_0$. 
\[ K_0 = \frac{1}{h} \left( \begin{array} {ccc}
1 & -1 & \\
-1 & 1 & \\
\vdots & \ddots & \ddots \\
\end{array} \right) \\
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1 \\
\end{array} \\
+ \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1 \\
\end{array} \\
+ \cdots \\
\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 2 & -1 \\
\vdots & \ddots & \ddots \\
0 & \vdots & \vdots & 0 \\
0 & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
\end{array} \right) \\
= \frac{1}{h} \left( \begin{array} {ccc}
1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
\end{array} \right) 
\]

as before.
Similarly we get

\[ K_1 = \frac{h}{6} \left[ \begin{array}{ccc}
2 & 1 & \\
1 & 2 & \\
0 & 0 & 0
\end{array} \right] + \left[ \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right] + \left[ \begin{array}{ccc}
0 & 0 & 2 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{array} \right] \]

\[ \frac{h}{6} \left[ \begin{array}{ccc}
2 & 1 & 0 \\
1 & 4 & 1 \\
0 & 0 & 0
\end{array} \right] \]

\[ \frac{h}{6} \left[ \begin{array}{ccc}
0 & & 0 \\
& & \ddots \\
0 & & 0
\end{array} \right] \]

\[ \left[ \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right] \]

as before.
• This is typical for the finite element method: Think about everything, and analyze everything, in terms of individual elements, and then put together the global systems. In this context the word “element” refers to one piece (interval in this case) of the domain.

• If the differential equation is linear then the system is linear.

• Otherwise the global system is nonlinear, but still sparse. In that case we might solve it by Newton’s Method and assemble the Jacobian using the assembly techniques.

• We are done with ODEs. Will need to discuss finite elements for PDEs.
Summary, ODEs

• IVPS $\rightarrow$ VSVO, stiff/non-stiff

• BVPs:
  – Shooting
  – Finite Differences
  – Finite Elements

• Principles:
  – Discretization
  – local, global truncation error
  – Stability
  – convergence
  – asymptotic analysis
  – error control

• Finite Elements
  – weak form of Euler Equation (Galerkin, Least Squares, Collocation)
  – Basis functions have small support, leading to sparse systems
  – Basis functions are in cardinal form, the coefficients are function and derivative values.
  – Do everything on one piece of the domain partition.