Math 5620 Spring 2019

Notes of 4/1/19

Review

• Variational Principles lead to DEs, their Euler Equations.

• The Euler equation involves derivatives of twice the degree of those occurring in the VP.

• The Ritz Method minimizes the variational principle over a finite dimensional subspace $S^h$, giving an approximating function $u^h$.

• Some boundary conditions, i.e., essential boundary conditions, need to be built into $S^h$.

• $S^h$ may be an affine rather than a linear subspace (in the case that we have inhomogeneous boundary conditions).

• The Ritz Method produces an approximating function, rather than just a vector of numbers.

• It finds that approximation in the subspace that is closest to the solution, measured in the energy norm

$$
\|v\|^2 = (Lv, v) = a(v, v).
$$

• For a linear operator $L$ we need to solve a linear system in the coefficients of the approximate solution.

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• The Ritz Method is a finite element method if
  — The basis functions have small support. This implies that the coefficient matrix is sparse in general, and tridiagonal in the particular case of piecewise linear functions (in one variable).
  — The basis functions are in cardinal form, i.e., the approximating function is of a form where the coefficients are function values (or perhaps derivative values) of the approximating function. We can thus think of the linear system as a specific finite difference method.

Quick Review of Interpolation

• Suppose $p$ is the polynomial of degree $n$ that interpolates a function $u$ at $n + 1$ knots:

$$p(x_i) = u(x_i), \quad i = 0, 1, \ldots, n$$

• Then we know from Math 5610 that

$$u(x) - p(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_n)}{(n + 1)!}u^{(n+1)}(\xi)$$

for some $\xi$ in the interval spanned by the knots $x_j$ and $x$. (1)
More Error Analysis

• A major point of minimizing the distance between the true and the approximating function is that it facilitates the computation of error bounds.

Since the approximating function is better than any alternative in $S^h$ it is better, in particular, than the interpolant of the true solution.

• For example, let $L$ the linear interpolant of $u$ in the interval $[x_n, x_{n+1}]$, i.e.,

$$L(x) = \frac{x_{n+1} - x}{h}u(x_n) + \frac{x - x_n}{h}u(x_{n+1})$$

where $x_{n+1} = x_n + h$.

• Then the error is given by

$$E(x) = u(x) - L(x) = \frac{(x - x_n)(x - x_{n+1})}{2!} u^{(2)}(\xi)$$

• Supposing that

$$|f''(x)| \leq M_2$$

for all $x$ this implies that

$$|E(x)| \leq \frac{h^2}{8} M_2$$

• The error in the Ritz approximation therefore also is $O(h^2)$.
Note that this gives us a handle on the **global truncation** error whereas the Taylor Series based approach that we have used so far only applies to the **local truncation error**.
Major Dichotomy

- Recall that in Math 5610 we used piecewise polynomial function for the interpolation of given function values.

- Here the function values are not given, they are computed by solving a global linear system.

However, the underlying linear spaces (continuous piecewise linear functions) are the same!

More finite elements

- How can we increase the accuracy?

- One idea, of course, is to decrease $h$.

- But we would like to increase the order of the method.

$C^0$ Quadratics

- For example we could use a quadratic on each interval, and determine that quadratic by interpolating at the endpoints and at the center.

- What kind of linear system would we get?
• The approximating function would be piecewise quadratic, but only continuous.

• The global Truncation error would be $O(h^3)$.

• There is an obvious generalization to polynomials of degree greater than 2.

• Note that we did not consider those spaces in Math 5610.
Piecewise Cubic Hermite

\[ H(x) = \sum_{i=0}^{N} u(x_i) H_i(x) + \sum_{i=0}^{N} u'(x_i) \bar{H}_i(x) \]

\[ H_i, \bar{H}_i \; \text{in} \; \{ s \in C[a,b] : s \; \text{cubic on each} \; [x_i, x_{i+1}] \} \]

\[ H_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \; H_i'(x_j) = 0 \]

\[ \bar{H}_i(x_j) = 0, \; \bar{H}_i'(x_j) = \delta_{i,j} \]

\[ x_i = a + ih, \quad i = 0, \ldots, N \]
Other Variational Techniques

• Our justification for the Ritz Method has been compelling: We get the best possible approximation in the approximating space.

• Instrumental in obtaining that property is the correspondence between the Euler Equation and the minimization of the associated variational principle.

• What if we have no VP. What if we just have the equation?

• Not all is lost.

• Suppose we have the differential equation

\[ Lu = f \]

and we want to approximate \( u \) with a linear combination of basis functions

\[ u(x) \approx u^h(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x). \]

• In this context, the \( \phi_i \) are called the trial functions

• Suppose we also have some test functions \( \psi_i, i = 1, \ldots, N. \)

• Then we can define the coefficients \( \alpha_j \), for example, by the system of equations

\[ (Lu^h, \psi_i) = (f, \psi_i), \quad i = 1, 2, \ldots, N \quad (2) \]
(2) is called the **weak form** of the variational principle minimization problem.

- In the case that $\psi_i = \phi_i$ and $Lu = f$ is the Euler equation for

$$I(u) = (Lu, u) - 2(f, u)$$

this is precisely the Ritz Method.

- But the weak form (2) makes sense even if there is no variational principle.

- It is even possible that the differential operator not be linear! If it is in fact linear, then (2) is a linear system, otherwise it is a nonlinear system.

- In the case that test and trial functions are the same,

$$\psi_i = \phi_i, \quad i = 1, \ldots, N$$

the method is called the **Galerkin Method**.

- But other choices of the test functions are possible.

- For example, in the **collocation method** the test function could be a **delta function**:

$$\psi_i(x) = \delta(x - z_i)$$

where the $z_i$ are suitably chosen **collocation points**.
The “delta function” is defined by
\[
\delta(x) = 0 \text{ if } x \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(x) = 1.
\]

This of course does not make sense, strictly speaking.

You can think of the delta function as the limit of a sequence of bell-shaped functions whose area remains constant and equal to 1, and whose support approaches zero.

Since
\[
\int_{0}^{\infty} \delta(x)f(x) = f(0)
\]
we get
\[
(Lu)(z_i) = f(z_i)
\]
and you can think of collocation as satisfying the DE \(LU = f\) at the collocation points. Essentially the delta function is used as a notation for (3).

Rarely are the collocation points equally spaced.

A common choice, for example, are the roots of the Chebychev polynomials. These lie in the interval \((-1, 1)\) and are given by
\[
z_i = \cos \frac{i}{N+1}, \quad i = 1, 2, \ldots, N
\]

Another possible choice is
\[
\psi_i = L\phi_i.
\]
• This is equivalent to solving the Least Squares problem

\[(Lu^h - f, Lu^h - f) = \min\]

(exercise).

• In the non-Ritz variational approaches the error analysis becomes more difficult but leads to very similar results: The global error is of the same order as the interpolation error.

• **Summary** of some salient points:
  
  – Finite elements are characterized by small local support giving rise to sparse (banded) systems.
  
  – Coefficients of the approximant are values of the approximant and its derivatives.
  
  – simplicity of the linear system is (usually) more important than the higher accuracy associated with smaller but higher order systems.
  
  – Setting the linear systems may require integrals that are not possible analytically. In those cases numerical integration is required.
  
  – Essential boundary conditions need to be built into the approximating space.
  
  – Error analysis goes back to polynomial interpolation.