IVPs of ODEs

- Initial Value Problems of Ordinary Differential Equations
- Some Classics:
• We will consider the (very general) first order system:

\[ y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b \]

where \( y, y_0, \) and \( f \) are vectors with \( m \) components.

• If \( m = 1 \) we have the usual kind of scalar IVP.

• When it matters (which will be rarely) we will use left superscripts for indicating components of a vector. Most of the time our system will be scalar \((m = 1)\) and we will omit the superscripts.

• (Subscripts will indicate terms of a sequence.)

• Example for \( m = 2 \)

\[
\begin{align*}
1y' &= \frac{1}{x} y + x 2y \\
2y' &= x \frac{2y^2 - 1}{y}
\end{align*}
\]

(1)

• As usual we get some integration constants that are determined by initial conditions. In this case we get (exercise)

\[
\begin{align*}
1y(x) &= \frac{x (\cos(c_1 x + c_2))}{c_1} \\
2y(x) &= -\sin(c_1 x + c_2)
\end{align*}
\]
• If the side conditions determining the constants are the **initial condition**

\[ y(a) = y_0 \]

then we have an **initial value problem**.

• A major concern is the **existence** and **uniqueness** of the solution.

• These are assured, for example, if \( f \) is **Lipschitz continuous** in \( y \) and differentiable in \( x \).

• Lipschitz Continuity means that there is a (Lipschitz) constant \( L \) such that

\[
\|f(x, y) - f(x, y^*)\| \leq L\|y - y^*\|.
\]

• If this is true for all \( x \) in \([a, b]\) and \( y, y^* \) in \( \mathbb{R}^m \) then a unique solution exists for all \( y_0 \).

• The constant \( L \) will make an appearance a couple of times in this course.

• We discussed **quadrature** where \( f \) does not depend on \( y \). That’s obviously Lipschitz continuous.

• If we have a system \((m > 1)\) and \( f \) is independent of \( y \), then we just have \( m \) separate scalar quadrature problems.

• The function \( f \) may be independent of \( x \). In that case we say that the ODE is **autonomous**.
• It is always possible to convert an ODE to an autonomous system.

• Just treat $x$ as a dependent variable and add the equation

\[ x' = 1, \quad x(a) = a. \]

• For example, in (1) we would replace $m$ with 3, $x$ with $3y$, and add the equation

\[ 3y' = 1, \quad 3y(a) = a. \]

• Thus we may assume, when it matters, that our system is autonomous. This will be useful when we discuss Runge-Kutta methods.

• Our IVP is of first order. Any higher order ODE can be converted to a first order system.

• For example, if

\[ y'' = f(x, y, y') \]  \hspace{1cm} (2)

we introduce a new variable $w \in \mathbb{R}^m$ and replace (2) with the first order system

\[ w' = f(x, y, w) \]
\[ y' = w \]

• This is a system of size $2m$, and we could proceed in an obvious manner for systems of order higher than 2.
However, the second order system

\[ y'' = f(x, y) \]

where \( f \) does not depend on \( y' \) is special. Such systems arise, for example, when analyzing the motion of objects in space under the influence of gravity. There are special purpose methods (and in fact even special purpose hardware) for these kinds of problems, but we will largely ignore them.

**Linear Systems**

- A **linear system** is of the form
  \[ f(x, y) = A(x)y + g(x) \]
  where \( A \) is an \( m \times m \) matrix and \( g \) is a given function.
- If \( A(x) \) is constant, then this is a **constant coefficient linear system**:
  \[ f(x, y) = Ay + g(x). \]
- A yet more special case is a **homogeneous constant coefficient linear system** where \( g(x) = 0 \) and
  \[ y' = Ay. \] (3)
- Such systems play a major role in applied mathematics.
Major Principle: The general solution of a (constant coefficient) linear problem is any particular solution, plus the general solution of the homogeneous problem.

- We saw this in linear algebra, the general solution of $Ax = b$ is any particular solution, plus the general solution of the homogeneous problem $Ay = 0$.

\[
\begin{align*}
Ax &= b \\
Az &= b \\
A(z - x) &= b \\
A(x - z) &= 0
\end{align*}
\]
Let’s verify this principle for

\[ y' = Ay + g(x). \]

\[ z' = Az + g(x) \]

\[ y' - z' = (y - z)' = A(y - z) \]

\[ w' = Aw \]

\[ w = z - y \]

\[ z' = Az + g(x) \]

\[ y' = Ay + g(x) \]

\[ z = y + w \]
• The solutions of the homogeneous system

\[ Ay = 0 \]

form a **linear space**: the set of solutions is closed under addition and scalar multiplication.

• Moreover, if \( y_0 \) is an eigenvector of \( A \), i.e.,

\[ Ay_0 = \lambda y_0 \]

then

\[ y(x) = C e^{\lambda x} y_0 \]

is a solution of (3) since

\[ y' = C \lambda e^{\lambda x} y_0 = \lambda (C e^{\lambda x} y_0) \]

\[ = \lambda Ae^{\lambda x} y_0 \]

\[ = Ay \]

• If all the eigenvalues are distinct then the corresponding \( m \) solutions of (3) are linearly independent and form a basis of the solution space.

• If the eigenvalues are repeated, or \( A \) is defective, then things more complicated, but we will ignore this issue (even though it is a fascinating subject).

Note, however, that the eigenvalues of \( A \) may be complex.
• In that case we have to take linear combinations to cancel the imaginary parts and obtain oscillating solutions.

\[ y(x) = A \cos x + B \sin x \]

\[ y'' = -y \]

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

\[ e^{-i\theta} = \cos \theta - i \sin \theta \]

\[ w = y' \]

\[ w' = -y \]

\[ y' = w \]

\[ \begin{bmatrix} w' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ i & 0 \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix} = \lambda \begin{bmatrix} w \\ y \end{bmatrix} \]

\[ |A \cdot 2i| = \left| \begin{bmatrix} 0 & -2i \\ i & 0 \end{bmatrix} \right| = x^2 + 1 \]

\[ \lambda = i, -i \]
eucectors \begin{bmatrix} i \\ i \end{bmatrix} \begin{bmatrix} -i \\ -i \end{bmatrix}