Boundary Value Problems

Shooting Methods

• Consider

\[ y'' = f(x, y, y'), \quad y(a) = A, \quad y(b) = B. \]

• The basic idea of shooting methods is to replace the above boundary problem with an initial value problem where one initial condition is unknown, and determined so as to satisfy the BVP.

\[ y'' = f(x, y, y'), \quad y(a) = A, \quad y'(a) = s. \]

• The solution of this IVP is a function \( y(x, s) \) where we want the parameter \( s \) to be such that

\[ \varphi(s) = y(b, s) - B = 0. \]

• This is a nonlinear equation in \( s \) and the methods we discussed for the solution of nonlinear equations are all applicable.
The Bisection Method

\[ f(b, s) \sim b \]
The Secant Method
Newton’s Method

- Newton’s Method?

\[ s_0 = \frac{B - A}{b - a} \quad \text{for example} \]

\[ s_{n+1} = s_n - \frac{\varphi(s_n)}{\varphi'(s_n)} = s_n - \frac{\gamma(b, s) - B}{\zeta(b, s)} \]

\[ \varphi(s) = \gamma(b, s) - B \]

\[ \gamma'' = f(x, y, y') \quad \gamma(a) = A \quad \gamma'(a) = s \]

\[ \gamma''(x, s) = f(x, y(x, s), y'(x, s)) \]

\[ \zeta(x, s) = \frac{\partial}{\partial s} \gamma(x, s) \]

\[ \zeta''(x, s) = f_y z + f_{yy} z' \quad \zeta(a, s) = 0 \]

\[ \zeta'(a, s) = 1 \]
Example:

\[ y'' = k^2 y, \quad y(-1) = y(1) = 1, \quad k \gg 1. \]

\[ y(x) = \alpha e^{kx} + \beta e^{-kx} \]
\[ y(1) = \alpha e^k + \beta e^{-k} = 1 \]
\[ y(-1) = \alpha e^{-k} + \beta e^k = 1 \]
\[ \alpha + \beta e^{-2k} = e^{-k} \]
\[ \alpha + \beta e^{2k} = e^k \]
\[ \beta(e^{2k} - e^{-2k}) = e^{k} - e^{-k} \]
\[ \beta = \frac{e^k - e^{-k}}{e^{2k} - e^{-2k}} \approx e^{-k} \]

shooting is hopeless
Multiple Shooting

break \[ a,b \] subintervals

\[
\begin{align*}
\gamma'(x) & \approx \frac{\gamma(x+h) - \gamma(x-h)}{2h} \\
\gamma''(x) & \approx \frac{\gamma'(x+h) - \gamma'(x-h)}{h^2} \\
& \approx \frac{\gamma(x+h) - 2\gamma(x) + \gamma(x-h)}{h^2}
\end{align*}
\]
Finite Differences

\[ y'' = f(x, y, y'), \quad y(a) = A, \quad y(b) = B \]

- Discretize:

\[ h = \frac{b - a}{N}, \quad x_n = a + nh, \quad y_n \approx y(x_n), \quad n = 0, 1, \ldots N \]

\[ \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = f\left( x_n, y_n, \frac{y_{n+1} - y_{n-1}}{2h} \right) \]

- Nonlinear system
- Tridiagonal
- Tradeoff between order and complexity of system
- Newton’s Method:
Increasingly accurate shooting
**Finite Elements**

- Differential Equation comes from a variational principle:

\[
I(y) = \int_{-1}^{1} y'^2 + k^2 y^2 \, dx = \min, \quad y(-1) = y(1) = 1.
\]