Zadunaisky’s Device

• Pedro Zadunaisky (1917–2009) was an Argentinian astronomer who studied non-gravitational forces acting on comets. The sunward side produces gases which exit the comet and provide a very small thrust. Because of the comet’s rotation and the varying direction of gas jets the total thrust is not in an obvious direction like directly away from the sun.

• Zadunaisky integrated the equations governing the gravitational effects on the comet and found some very small deviations from the actual orbit. He needed to rule out the possibility that those deviations were simply (very small) errors in his numerical solution.

• To that end he came up with the following idea:

  1. Solve the original problem

  2. Use the numerical solution to set up a closely related pseudo-problem with known exact solution

  3. Solve the pseudo-problem using the same numerical method as that used for the original problem.
4. Compute the global error in the solution of the pseudo-problem, and use it as an estimate of the global error in the solution of the original problem.

- Zadunaisky presented his work at Astronomy conferences. Later, mathematicians took notice and analyzed his idea in more depth. They refer to the technique as Iterated Defect Correction. A host of papers and at least one book have been written on the subject. Some basic references include:


- **Assumptions.** The statements in this note
are true only if certain assumptions about the underlying initial and boundary value problems are satisfied. In particularly the functions involved need to be sufficiently often differentiable. The numerical methods have to satisfy some more serious assumptions about the expansions of global truncation errors. The particular methods described in this note do satisfy those assumptions.

**Initial Value Problems**

- Consider the original (scalar initial value) problem

\[ y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b. \tag{1} \]

We assume as usual that

\[ x_n = a + nh \quad \text{where} \quad h = (b - a) / N \tag{2} \]

and we construct a sequence of approximation

\[ y_n \approx y(x_n), \quad n = 0, 1, \ldots, N. \tag{3} \]

Let \( p(x) \) be the interpolating polynomial of degree \( N \), i.e.,

\[ p(x_n) = y_n, \quad n = 0, 1, \ldots, N \tag{4} \]

Then consider the pseudo-problem

\[ y' = f(x, y) + p'(x) - f(x, p(x)), \quad y(a) = y_0 = p(a). \tag{5} \]
Clearly, the exact solution of (5) is \( y = p(x) \). Now solve (5) by the same method as used for the solution of (1), giving the numerical solution

\[
    z_n \approx p(x_n), \quad n = 0, 1, \ldots, N.  \quad (6)
\]

The differential equation in (5) is very close to that in (1), and so the errors should be very similar. We thus approximate the global error in \( y_n \) by the computable global error in \( z_n \):

\[
e_n = y(x_n) - y_n \approx p(x_n) - z_n = y_n - z_n, \quad n = 0, 1, \ldots, N.
\]

It turns out that this estimate can be extraordinarily effective.

- **Exercise 1.** What do you think about using the pseudo-problem

\[
y' = f(x, p(x)) + p'(x) - f(x, y), \quad y(a) = y_0 = p(a)?  \quad (7)
\]

which has the same solution as (5)?

- Various numerical methods may be employed to construct the sequences \( \{y_n\} \) and \( \{z_n\} \). For the sake of being specific let’s follow Zadunaisky and use the standard fourth order Runge-Kutta method:
\[ y_{n+1} = y_n + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4) \]

\[ K_1 = f (x_n, y_n) \]

\[ K_2 = f \left( x_n + \frac{h}{2}, y_n + \frac{h}{2} K_1 \right) \] \hspace{1cm} (8)

\[ K_3 = f \left( x_n + \frac{h}{2}, y_n + \frac{h}{2} K_2 \right) \]

\[ K_4 = f (x_n + h, y_n + hK_3) \]

• Of course we do not contemplate using interpolating polynomials of a very high degree. Instead we settle on a low polynomial degree \( d \), say, and apply the idea piecewise. There are two variations:

• The non-integrated approach. Solve the original problem on the entire interval. Then partition \([a, b]\) into subintervals \([x_{id}, x_{(i+1)d}]\), \( i = 0, 1, \ldots \), each containing \( d + 1 \) points. On each subinterval construct the interpolating polynomial. Then solve the pseudo-problem. Note that the piecewise polynomial interpolant is only continuous. This is no problem since we are using a one-step method.

• The integrated approach. After processing the first interval, use the error estimate to improve the numerical solution and start in the next interval from the improved numerical solution at the right endpoint of the preceding interval. Repeat as needed.
• Iterated Application. Zadunaisky’s device can be repeated. We use the error estimate to improve the numerical solution. Then we interpolate the improved solution and construct a new pseudo-problem. Let $y_{n}^{[0]}$ be the solution of the original problem. Then for $i = 0, \ldots, s - 1$, say, do the following: Let $p^{k}$ be the interpolant of $y_{n}^{k}$. Solve the pseudo-problem

\begin{equation}
y' = f(x, p^{k}(x)) + p^{k'}(x) - f(x, y), \quad y(a) = y_{0} = p^{k}(a) \tag{9}
\end{equation}

Denote the solution of the (9) by $z_{n}^{[k]}$. Let

\begin{equation}
y_{n}^{[k+1]} = y_{n}^{[0]} + y_{n}^{[k]} - z_{n}^{[k]} \tag{10}
\end{equation}

• The following Theorem covers the case $s = 1$ and the non-integrated approach:

• Theorem 2. Suppose the basic method is a Runge-Kutta method of order $p$ and suppose that $p + 1 \leq d \leq 2p$. Then there exists a constant $C$ which is independent of $h$ such that

\begin{equation}
\max_{n} |(y(x_{n}) - y_{n}) - (y_{n} - z_{n})| \leq C h^{d} \tag{11}
\end{equation}

If $d > 2p$ then there exists such a constant such that

\begin{equation}
\max_{n} |(y(x_{n}) - y_{n}) - (y_{n} - z_{n})| \leq C h^{2p} \tag{12}
\end{equation}
In other words, the order of the Runge-Kutta method can be doubled with Zadunaisky’s device.

- This is sensational!
- We have discussed at some length how to estimate local errors. All of our estimates have been based on comparing two approximations, and amounted to an asymptotically estimation of the leading term of the local truncation error. Zadunaisky’s device provides an asymptotically exact estimate of the leading $p$ terms of the global truncation error.
- One way of looking at this is that by using the error estimate to improve the numerical approximation one can double the order of the Runge-Kutta method by doubling the number of function evaluations. A Runge-Kutta method with twice the order would require more than twice as many function evaluations.
- On the other hand, we do get a very accurate error estimate which is what Zadunaisky needed to rule out artifacts of the numerical method.
- Interpolating Derivatives. A modification of Zadunaisky’s device is based on interpolating the values $f(x_n, y_n)$ and then constructing $p$ by integration.
- Zadunaisky’s idea can also be applied to the boundary value problem
\[ y'' = f(x, y), \quad y(a) = A, \quad y(b) = B. \quad (13) \]

- For details consult the literature.