Math 5610 Fall 2021

Notes of 11/16/21

Announcements

• Wednesday and Friday: Discussion of Term Project.

• We’ll meet on Zoom. Link on Canvas, password is letmein

• for reference, link is

  https://utah.zoom.us/j/96459993308

• Presentations are in the sequence in which submissions happen to be listed in my records:

  • Wednesday:
    – 10:45 Zach Candelaria
    – 10:50 Elijah Kirat
    – 10:55 Liz Maynard and Morgan Kelley
    – 11:00 Matthew Roser
    – 11:05 Eli Counterman and Taos Transue
    – 11:10 Corrin Krogh, Matthew Fontaine, and Genesee Nelson
    – 11:15 Colin McNab
    – 11:20 Brady Tan
    – 11:25 Taito Fuji and Edward Gu
    – 11:30 general discussion
• Friday
  – 10:45 Carolyn Laprete, Tayyaba Chaudry, and Bobbi Covington
  – 10:50 David Jones, Preston Malen, and Lela Feaster
  – 10:55 Cade Madson
  – 11:00 Apoorva Pedgaonkar, Kaden Dvorak, and Bao Le
  – 11:05 Emily Toney and Bo Caldwell
  – 11:10 Valery Kennion and Donald Harrison
  – 11:15 Julia Ma, Saleema Qazi, and Savannah Simmons
  – 11:20 Ben Perry
  – 11:25 Wrapup and general discussion

Attendance of both sessions is required!

• Please let me know of any typos in your name, or if your name is missing from the above list.

• You will be able to share your screen.

• Your goal is to make your presentation as interesting and instructive for your classmates as you can.

• Be prepared to answer questions!

• I will put your report on Canvas, unless you send me some other material you’d like to put there.
The Weierstrass Approximation Theorem

There should be a technical proof in every math course. I encourage you to study this proof and to make sure you understand every detail. Once you do you will have greatly increased your technical abilities. You will also know why the name of Bernstein is in the Bernstein-Bézier form of a polynomial. Let me know if you have any questions.

Polynomials are widely used. Reasons for their utility include the fact that they are easily evaluated, differentiated, integrated, and patched together. Another major reason is the fact that they are dense in $C[a,b]$. This is the contents of the celebrated

**Theorem (Weierstrass Approximation Theorem) (1885).** Let $f \in C[a,b]$. Given any $\epsilon > 0$ there exists a polynomial $p_n$ of sufficiently high degree $n$ for which

$$|f(x) - p_n(x)| \leq \epsilon \quad \text{for all } x \in [a,b]. \quad (1)$$

Many proofs of the Weierstrass Theorem are known. In these notes we present one due to Bernstein, because it is constructive (it actually tells what that polynomial is), and it gives an introduction to an extremely useful form of a polynomial, the *Bernstein-Bézier form* of a polynomial (which we will discuss in depth later in this class).

For details of the proof see, e.g., the classic

Before starting, notice the following:

- $f$ only needs to be continuous. We saw an error expression for an interpolant of $f$ that involved high order derivatives of $f$. The Weierstrass Theorem is much more general than that.

- It is essential that the domain $[a, b]$ be closed and bounded. Easy examples (exercise!) show that the assumptions of the Theorem are necessary.

- While we do not require that $f$ be differentiable, it can be shown (see Davis) that if $f$ is differentiable then the Bernstein approximation given below approximates the function and its derivatives simultaneously. Derivatives of the approximation converge to the corresponding derivatives of $f$ as $n$ goes to infinity.

Without loss of generality, let

$$[a, b] = [0, 1].$$

(2)

(For any other interval we’d use a linear change of variables which preserves polynomials and their degrees. Easy exercise!)

The $n$-th Bernstein Polynomial for $f$ is defined to be

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad (3) \quad n \geq 0$$
\[
\frac{d}{dx} x^k (1-x)^{n-k} = k x^{k-1} (1-x)^{n-k-1} - x^{k-1} (n-k) (1-x)^{n-k-2} = 0
\]

\[
x^{k-1} (1-x)^{n-k-1} \left( k (1-x) - x (n-k) \right) = 0
\]

\[
k - kx - nx + kx = 0
\]

\[
k - nx = 0
\]

\[
x = \frac{k}{n}
\]
where, as usual
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
and
\[
n! = \begin{cases} 1 & \text{if } k = 0 \\ 1 \times 2 \times 3 \ldots \times n & \text{otherwise} \end{cases}
\] (4)

\(B_n\) is called the Bernstein Operator. The binomial coefficient \(\binom{n}{k}\) is pronounced \(n\ choose\ k\).

It’s easy to check that the monomials \(x^k(1-x)^{n-k}\) are non-negative in \([0, 1]\) and assume their maximum at \(x = \frac{k}{n}\).

Figure 1 shows graphs of the Bernstein Basis functions \(b_k = \binom{n}{k} x^k (1-x)^{n-k}\) for \(n = 10\) and \(k = 0, \ldots, 10\).

Figure 2 shows how the basis (very slowly) functions become more pointed as the polynomial degree increases. (These graphs were computed in 8,000 digit arithmetic).

The following statement was proved by Bernstein. The Weierstrass Theorem is a corollary of this result.

**Theorem (Bernstein).** If \(f \in C[0, 1]\) then
\[
\lim_{n \to \infty} B_n(f, x) = f(x),
\] (5)
uniformly for \(x \in [0, 1]\).

The phrase *uniformly* in this statement means that in order to make \(|f(x) - B_n(f, x)|\) less than \(\epsilon\) one can choose \(n > N(\epsilon)\) where \(N(\epsilon)\) is independent of \(x\).
To begin with, we rewrite

$$B_n(f, x) = \sum_{t=0}^{n} \triangle^t f(0) \binom{n}{t} x^t$$  \hspace{1cm} (6)
Figure 2. Graphs of $2^{2N} x^N (1 - x)^N$, $N = 2^n, n = 0, \ldots, 10$.

where

$$\triangle^0 f(x) = f(x)$$
$$\triangle^{t+1} f(x) = \triangle^t f \left( x + \frac{1}{n} \right) - \triangle^t f(x) \quad t = 0, 1, 2, \ldots$$

(7)

$\triangle$ is called a forward difference operator. We obtain, for example,
\[ \triangle^0 f(0) = f(0) \]
\[ \triangle^1 f(0) = f\left(\frac{1}{n}\right) - f(0) \]
\[ \triangle^2 f(0) = \triangle^1 f\left(\frac{1}{n}\right) - \triangle^1 f(0) \]
\[ = f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right) - \left( f\left(\frac{1}{n}\right) - f(0) \right) \]
\[ = f\left(\frac{2}{n}\right) - 2f\left(\frac{1}{n}\right) + f(0) \]

In general (exercise, use induction!)

\[ \triangle^t f(0) = \sum_{k=0}^{t} f\left(\frac{k}{n}\right) \binom{t}{k} (-1)^{t-k}. \] (9)

The proof of (6) is elementary, but a little
technical. We write

\[ B_n(f, x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k} \]

by (3)

\[ = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} x^{n-k-j} \]

by the Binomial Theorem

by reorganizing

\[ = \sum_{k=0}^{n} \sum_{j=0}^{n-k} f \left( \frac{k}{n} \right) \binom{n}{k} \binom{n-k}{j} (-1)^{n-k-j} x^{n-k-j} \]

We now change summation indices, by replacing \( n-j \) with \( t \). Thus as \( j \) runs from 0 to \( n-k \), \( t \) runs from \( k \) to \( n \).

We obtain

\[ B_n(f, x) = \sum_{t=0}^{n} x^t \sum_{k=0}^{t} f \left( \frac{k}{n} \right) \binom{n}{k} \binom{n-k}{n-t} (-1)^{t-k} x^t \]

\[ = \sum_{t=0}^{n} x^t \sum_{k=0}^{t} f \left( \frac{k}{n} \right) \binom{n}{k} \binom{n-k}{n-t} (-1)^{t-k} \cdot \]

(11)

since, for any numbers \( a_{tk} \)

\[ \sum_{k=0}^{n} \sum_{t=0}^{n} a_{tk} = \sum_{t=0}^{n} \sum_{k=0}^{t} a_{tk} \cdot \]

(12)
\[ x^k (1-x)^{n-k} = x^\sum_{j=0}^{k} \frac{k}{j} (-1)^j (n-k-j) \times (n-k-j) \times (n-j) \]

\[ = \sum_{j=0}^{k} \frac{k}{j} (-1)^j (n-k-j) \times (n-j) \]

\[ = \sum_{j=0}^{k} \frac{k}{j} (-1)^j (n-k-j) \times (n-j) \]
\[ (-x)^{n-k} = \sum_{j=0}^{n-k} \binom{n-k}{j} (-x)^{n-k-j} \]

\[ = \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j x^{n-k-j} \]

\[ = \sum_{j=0}^{n-k} (-1)^j x^{n-k-j} \]
Using
\[
\binom{n}{k} \binom{n-k}{n-t} = \frac{n!}{(n-t)!(t-k)!k!} = \binom{n}{t} \binom{t}{k},
\]
we get
\[
B_n(f, x) = \sum_{t=0}^{n} x^t \sum_{k=0}^{t} f\left(\frac{k}{n}\right) \binom{n}{k} \binom{n-k}{n-t} (-1)^{t-k}
\]
\[
= \sum_{t=0}^{n} x^t \binom{n}{t} \sum_{k=0}^{t} f\left(\frac{k}{n}\right) \binom{t}{k} (-1)^{t-k}
\]
\[
= \sum_{t=0}^{n} x^t \binom{n}{t} \triangle^t f(0) x^t
\]
as required.

Note that if \( f \) is a polynomial of degree \( m \) then since \( \triangle^t f(0) = 0 \) for \( t > m \) (exercise) \( B_n(f, x) \) is also a polynomial of degree \( m \) as long as \( n > m \).

This might lead one to suppose that the Bernstein operator \( B_n \) reproduces polynomials. This is not true in general. However, we obtain:

\[
B_n(1, x) = 1 \binom{n}{0} x^0 = 1,
\]
\[
B_n(x, x) = 0 \binom{n}{0} x^0 + \frac{1}{n} \binom{n}{1} x^1 = x,
\]
and

\[ B_n(x^2, x) = 0 \times \binom{n}{0} x^0 + \left( \frac{1}{n^2} - 0 \right) \binom{n}{1} x^1 + \left( \frac{4}{n^2} - \frac{2}{n^2} + 0 \right) \binom{n}{2} x^2 \]

\[ = x^2 + \frac{1}{n} (x - x^2) \]

(17)

So Bernstein Polynomials reproduce linear functions exactly, but not higher degree polynomials.

We want to show that for \( n \) sufficiently large \(|B_n(f, x) - f(x)| < \epsilon\). To get a handle on this we use the fact that

\[ f(x) = f(x) \times 1 \]

\[ = f(x)(x + (1 - x))^n \]

\[ = f(x) \times \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \]

(18)

and break the sum in \( f(x) - B_n(f, x) \) into two pieces:

\[ f(x) - B_n(f, x) = \sum_{k=0}^{n} \left( f(x) - f \left( \frac{k}{n} \right) \right) \binom{n}{k} x^k (1-x)^{n-k} = A + B \]

(19)
where

\[ A = \sum_{\left| \frac{k}{n} - x \right| \geq \delta} \left( f(x) - f \left( \frac{k}{n} \right) \right) \binom{n}{k} x^k (1 - x)^{n-k} \]

and

\[ B = \sum_{\left| \frac{k}{n} - x \right| < \delta} \left( f(x) - f \left( \frac{k}{n} \right) \right) \binom{n}{k} x^k (1 - x)^{n-k}. \]

(20)

We will bound both terms in (20) and then choose \( \delta \) such that each term is bounded by \( \epsilon / 2 \).

To bound the first sum we need more ingredients. Note that

\[ \left| \frac{k}{n} - x \right| \geq \delta \implies \frac{1}{\delta^2} \left( \frac{k}{n} - x \right)^2 \geq 1 \]  

(21)

and, by (15), (16), and (17), we get (exercise)

\[ \sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1 - x)^{n-k} = \]

\[ = B_n((x^2), x) - 2xB_n(x, x) + x^2B_n(1, x) \]

\[ = x^2 + \frac{1}{n}(x - x^2) - 2x^2 + x^2 \]

\[ = \frac{x(1 - x)}{n}. \]

(22)
We now obtain

\[
\sum_{|\frac{k}{n} - x| \geq \delta} \binom{n}{k} x^k (1 - x)^{n-k} \geq \delta \leq \frac{1}{\delta^2} \sum_{|\frac{k}{n} - x| \geq \delta} \left( \frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1 - x)^{n-k}
\]

by (21)

\[
\leq \frac{1}{\delta^2} \sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1 - x)^{n-k}
\]

by audaciously including more terms

\[
= \frac{x(1-x)}{\delta^2 n}
\]

by (22)

\[
\leq \frac{1}{4n\delta^2} \quad \text{(since } x(1-x) \leq \frac{1}{4})
\]

(23)

Next let's consider \( f \). It is continuous and thus bounded on \([0, 1]\). Suppose that \( M \) is such that

\[
|f(x)| \leq M \quad \text{and} \quad |f(x) - f(y)| \leq 2M \quad \text{for all } x, y \in [0, 1].
\]

(24)

Moreover, because of the continuity of \( f \) we can find \( \delta \) such that

\[
|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \frac{\epsilon}{2}.
\]

(25)

Using (20), choosing \( \delta \) such that (25) holds,
and $n$ such that

$$A \leq \frac{2M}{4n\delta^2} \leq \frac{\epsilon}{2} \quad (26)$$

holds, we obtain, finally

$$|f(x) - B_n(f, x)| = \left| \sum_{k=0}^{n} \left( f(x) - f \left( \frac{k}{n} \right) \right) \binom{n}{k} x^k (1 - x)^{n-k} \right|$$

$$\leq A + B$$

$$= \left| \sum_{|k/n-x|<\delta} \left( f(x) - f \left( \frac{k}{n} \right) \right) \binom{n}{k} x^k (1 - x)^{n-k} \right| +$$

$$+ \left| \sum_{|k/n-x|\geq\delta} \left( f(x) - f \left( \frac{k}{n} \right) \right) \binom{n}{k} x^k (1 - x)^{n-k} \right|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon \quad (27)$$

which is what we want to show.

Note that (26) implies that

$$n \geq \frac{M}{\epsilon \delta^2} \quad (28)$$
Assuming, for the sake of the argument, that $M = 1$ and $\epsilon = \delta = 10^{-6}$, which would be reasonable, for example, for an actual approximation of the sine function, we get that

$$n > 10^{18}$$

which of course is vastly too conservative, given that a Taylor Series of degree 12 would have an accuracy better than $10^{-6}$. 