Bivariate Interpolation

- Interpolation still means exact reproduction of data.
- The word **bivariate** refers to having two independent variables.
- Also common are the words **trivariate** (three independent variables), **multivariate** (more than one independent variable), and **univariate** (one independent variable).
- Much of what we’ll do in the bivariate context carries over to the more general multivariate context.
- However, there is a vast difference between univariate and multivariate contexts.
- We’ll consider five different kinds of interpolation:
  - **Tensor Product**
  - **Polynomial**
  - **Transfinite**
  - piecewise polynomial on triangles
  - meshless, or radial basis functions.
Tensor Product

- The phrase **tensor product** refers to a situation where the data lie on a rectangular grid, and we basically do univariate interpolation twice.

- Suppose we are given data

\[(x_i, y_j, f(x_i, y_j)), \quad i = 0, \ldots, m, \quad j = 0, \ldots, n\]

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grid lines parallel to the coordinate axis
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• We want an interpolant \( p \) satisfying

\[
p(x_i, y_j) = f(x_i, y_j)
\]

for all \( i \) and \( j \).

• An important basic idea is to consider operators that do the job in one of the variables and then compose, or otherwise combine them.

• Consider polynomial interpolation. Define

\[
\begin{align*}
    P_x f(x, y) &= \sum_{i=0}^{m} f(x_i, y) L_i(x) \quad \text{where} \quad L_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}. \\
    P_x f(x_k, y) &= \sum_{i=0}^{m} f(x_i, y) L_i(x_k) = f(x_k, y) \quad \text{where} \quad L_i(x_k) = \delta_{i,k}. \\
    P_y f(x, y) &= \sum_{j=0}^{n} f(x, y_j) M_j(x) \quad \text{where} \quad M_j(y) = \frac{\prod_{i \neq j}(y - y_i)}{\prod_{i \neq j}(y_j - y_i)}. \\
    P_y(x_k, y) &= f(x_k, y) \\
    P_x &= \frac{\partial}{\partial x} p
\end{align*}
\]
• The overall interpolant is then simply the composition of those two interpolants.

\[ Pf = P_x P_y f \]

\[ = P_x \sum_{j=0}^{n} f(x, y_j) M_j(y) \]

\[ = \sum_{i=0}^{m} \left( \sum_{j=0}^{n} f(x_i, y_j) M_j(y) \right) L_i(x) \]

\[ = \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i, y_j) M_j(y) L_i(x) \]

\[ = p(x, y) \]

\[ P \left( x_{i}, y_{j} \right) = \sum \sum f(x_{i}, y_{j}) M_{j}(y_{j}) L_{i}(x_{i}) = f(x_{i}, y_{j}) \]

• Clearly, \( p \) interpolates at all points \((x_i, y_j)\).
**Bilinear Interpolation**

- For simplicity, consider the unit square.
- Let
  \[ P_x f(x, y) = (1 - x)f(0, y) + xf(1, y) \]
  and
  \[ P_y f(x, y) = (1 - y)f(x, 0) + yf(x, 1) \] (1)

- Clearly \( P_x \) interpolates at all points \((0, y)\) and \( P_y \) interpolates all points \((x, 0)\) and \((x, 1)\).
- The composition gives the **bilinear** interpolant to the function values at the vertices of the unit square.

\[
P f(x, y) = P_x P_y f(x, y) \\
= P_x ((1 - y)f(x, 0) + yf(x, 1)) \\
= (1 - x)((1 - y)f(0, 0) + yf(0, 1)) \\
+ x((1 - y)f(1, 0) + yf(1, 1)) \\
= (1 - x)(1 - y)f(0, 0) \\
+ (1 - x)yf(0, 1) \\
+ x(1 - y)f(1, 0) \\
+ xyf(1, 1) \\
= A + Bx + Cy + Dxy
\]

- The interpolant is **bilinear** which means that setting one variable to a constant gives a function that is linear in the other variable.
• Note that the restriction of the bilinear function to a line, such as $y = x$, in general gives a \textit{quadratic} function.
Transfinite Interpolation

• We illustrate the idea by interpolating everywhere on the boundary of the unit square.

• This is not as far fetched as it may sound. For example, the data could come from a grid that approximate the shape of a vehicle such as a car or aircraft. The curves along the grid lines could have been designed by some univariate scheme. Or we might have a vehicle, such as a survey ship, move along a line and collect data densely along the line.

• Let $P_x$ and $P_y$ be the linear operators defined in (1).

• The following bilinearly blended Coon’s patch interpolates on the boundary of the unit square. It is also called the Boolean Sum of the operators $P_x$ and $P_y$.

$$Qf = P_x f + P_y f - P_x P_y f$$

$$= (1 - x)f(0, y) + xf(1, y)$$

$$+ (1 - y)f(x, 0) + yf(x, 1)$$

$$- (1 - x)(1 - y)f(0, 0)$$

$$- (1 - x)yf(0, 1)$$

$$- x(1 - y)f(1, 0)$$

$$- xyf(1, 1)$$

• Check interpolation
Exercise: try cubic Hermite interpolation on the unit square, finite and transfinite.
Scattered Data

- The phrase “scattered data” means that the data sites do not form a rectangular grid.
- So we are given data
  \[(x_i, y_i, z_i), \quad i = 1, 2, \ldots, n\]
  and we want an interpolant \(P\) such that
  \[P(x_i, y_i) = z_i, \quad i = 1, 2, \ldots, n.\]
- The natural first step would be to attempt interpolation by a polynomial.
- The immediate stumbling block is that there is no unique polynomial associated with the number of data.
- A bivariate polynomial \(p\) of degree \(d\) is defined by
  \[p(x, y) = \sum_{i+j \leq d} \alpha_{ij}x^iy^j\]

\[
\begin{align*}
  d = 0 & \quad p = p_0 = \alpha_{00} & 1 \\
  d = 1 & \quad p_1 = p_0 + \alpha_{10}x + \alpha_{01}y = \alpha_{00} + \alpha_{10}x + \alpha_{01}y & 3 \\
  \quad 2 & \quad p_2 = p_1 + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 & 6 \\
  \quad 3 & \quad p_3 = p_2 + \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3 & 16
\end{align*}
\]
\[
\# \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}
\]

• How many parameters does a polynomial of degree \(d\) have?

\[
\text{poly} \ deg. \ d, \ k \ variables
\]

\[
\binom{n+k}{k} \text{ exercise!}
\]
Linear Interpolation

\((x_1, y_1)\) \((x_2, y_2)\) \((x_3, y_3)\)
Quadratic Interpolation

\[
\left( x_i, y_j, z_j \right)
\]

\[
\begin{vmatrix}
1 & x_1 & x_1^2 & x_1 y_1 & y_1^2 \\
1 & x_2 & x_2^2 & x_2 y_2 & y_2^2 \\
1 & x & x^2 & xy & y^2 \\
\end{vmatrix}
\]

\[
det V = A + Bx + Cy + Dx^2 + Ey + Fy^2
\]
• Maybe we shouldn’t use polynomials. Maybe there are better functions.

• Let

\[ p(x, y) = \sum_{i=1}^{n} \alpha_i \phi_i(x, y). \]

• How do we pick the basis functions \( \phi_i \)?

\[
\begin{bmatrix}
\phi_i(x_i, y_i) \\
\phi_j(x_j, y_j) \\
\vdots
\end{bmatrix}
\]

It turns out that no matter how we pick them, there are Lagrange Interpolation problems with distinct data sites such that the Vandermonde matrix is singular.