Wavelets

• Again, today we will see only the proverbial tip of the iceberg.

• Quick Review of Fourier Series:
  – Suppose $f$ is $2\pi$-periodic.
  – Then

$$f(x) \approx F_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

  – The coefficients are determined by

$$\int_{-\pi}^{\pi} (f(x) - F_n(x))^2 \, dx = \min$$

  – We saw that the basis functions are orthogonal and that

$$a_k = \int_{-\pi}^{\pi} f(x) \cos x \, dx$$
$$b_k = \int_{-\pi}^{\pi} f(x) \sin x \, dx$$

• A significant drawback of Fourier series is that any change in an arbitrarily small subinterval
of $[-\pi, \pi]$ affects all the coefficients of the Fourier Series.

- This can be overcome in many ways, including modifications of the Fourier Series.
- However, the most effective way of handling this, with many additional useful properties, is based on wavelets.
- The best introduction, in my opinion, to wavelets, is still
- Daubechies is one of the primary pioneers and developers of wavelets. Check the wikipedia for more information on her many distinctions and accomplishments.
- We’ll just look at one example, the subject of the first of Daubechies’ lectures, which, however, will introduce most of the key ideas.

The Haar Wavelet

- Wavelets are translations and dilations of a mother wavelet. In the special case of the Haar wavelet introduced by Alfred Haar, 1885–1933, in 1909

$$
\psi(x) = \begin{cases} 
1 & \text{if } 0 \leq x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} \leq x < 1 \\
0 & \text{else}
\end{cases}
$$

- introduced by Alfred Haar, 1885–1933, in 1909
Figure 1. The Haar Wavelet.

- The mother wavelet plays essentially the same role as the sine (or cosine) function in Fourier series.

- Note that the Haar wavelet has local support.

Translation and Dilation

- For integers \( m \) and \( n \) we define a Haar wavelet to be the function

\[
\psi_{m,n}(x) = 2^{-\frac{m}{2}} \psi(2^{-m}x - n).
\]

- Note that the support of \( \psi_{m,n} \) is the interval \([2^m n, 2^m(n + 1)]\). The length of the support is \(2^m\).

- A major property of Haar wavelets is that they are orthonormal, i.e.,

\[
\int_{\mathbb{R}} \psi_{m,n} \psi_{m',n'} = \begin{cases} 1 & \text{if } m = m' \text{ and } n = n' \\ 0 & \text{else} \end{cases}
\]
• This is easy to see. We first observe that

\[\int_{\mathbb{R}} \psi_{m,n}^2 = \int_{n2^m}^{(n+1)2^m} 2^{-m} \, dx = 2^{-m} 2^m = 1.\]

Now assume that \( n \neq n' \) and \( m = m' \). Then the supports of \( \psi_{m',n'} \) and \( \psi_{m,n} \) do not overlap and

\[\int_{\mathbb{R}} \psi_{m,n} \psi_{m',n'} = 0.\]

Finally, suppose that \( m \neq m' \), and without loss of generality suppose that \( m < m' \). Then the support of \( \psi_{m,n} \) lies entirely in a region where \( \psi_{m',n'} \) is constant. Since \( \int_{\mathbb{R}} \psi_{m,n} = 0 \) we get that

\[\int_{\mathbb{R}} \psi_{m,n} \psi_{m',n'} = 0.\]

**Approximation of an \( L_2 \) function**

• Suppose \( f \) is a function defined on the real line and satisfying

\[\int_{\mathbb{R}} f^2 < \infty\]

• We contemplate approximating \( f \) by a function \( g \) in the 2-norm:

\[\|f\| = \sqrt{\int_{\mathbb{R}} f^2}\]
Any $L_2$ function $f$, i.e., any function $f$ that satisfies
\[ \int_{\mathbb{R}} f^2 < \infty \]
can be arbitrarily well approximated in the sense that for all $\epsilon > 0$ there exists a finite linear combination $A_\epsilon$ of Haar wavelets such that
\[ \int_{\mathbb{R}} (f - A_\epsilon)^2 < \epsilon \]

- This fact is highly counterintuitive (in my opinion) since the integral of any linear combination of the Haar wavelets is zero, whereas the integral of $f$ may not be zero.

- But let us press ahead.

- First observe that any $L_2$ function $f$ can be arbitrarily well approximated by a function $f_0$ which is constant on each interval $[\ell 2^{-J_0}, (\ell + 1)2^{-J_0}]$ and whose support is $[-2^{J_1}, 2^{J_1}]$ where $J_0$ and $J_1$ are sufficiently large.

- We now come to the key idea. We write
\[ f^0 = f^1 + \delta_1 \]
\[ \delta_i = f - f' = f - \delta_1 - \delta_2 \]

where $f^1$ is an approximation of $f^0$ that is piecewise constant on subintervals of twice the size of those on which $f^0$ is constant.
• the value of $f^1$ on one of the new, twice as large, intervals is obtained by averaging the values of the original two subintervals.

• Note that $\delta^1 = f^1 - f^0$ is a linear combination of Haar Wavelets!

$$f^0 = f^1 + \sum_{\ell} c_{-J_0+1,\ell} \psi_{-J_0+1,\ell}.$$ 

• Naturally, we apply the same trick to $f^1$, etc.

• Eventually we get

$$f^0 = f^{J_0+J_1} + \sum_{m=-J_0+1}^{J_1} \sum_{\ell} c_{m,\ell} \psi_{m,\ell}.$$ 

• $f^{J_0+J_1}$ consists of two constant pieces on the original support.

• $f^{J_0+J_1+1}$ consists of one constant piece on the original support.

• However, we continue, just widening the support.

• At every stage we average a constant with zero, halve the function value, and double the support.
In due course we obtain

\[ f^0 - \sum_{m=-J_0+1}^{J_1+K} \sum_{\ell} c_{m,\ell} \psi_{m,\ell} = f^{J_0+J_1+K} \]

So we have written our original function as a linear combination of Haar Wavelets + a remainder that is arbitrarily small.

Notice that we have sets of wavelets for every length scale (corresponding to a fixed value of \(m\)).

This is referred to as **multiresolution analysis**

All basis functions have local support.

A local phenomenon affects only the coefficients of those basis functions that are non-zero at the location of the phenomenon.

We considered the Haar wavelet for simplicity. It’s not unreasonable because many technological phenomena are in fact discrete and piecewise constant.

However, we can of course use different types of basis functions, particularly smoother ones. For examples, see Figure 1.8 on page 15 of Daubechies’ book. It is similar to Figure 2−2−.

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from https://www.intechopen.com/source/html/ 49109/media/image8.png
Figure 2. Some Orthogonal Wavelets.