Math 5610 Fall 2021

Notes of 11/10/2021

Wavelets

- Again, today we will see only the proverbial tip of the iceberg.
- Quick Review of Fourier Series:
  - Suppose \( f \) is \( 2\pi \)-periodic.
  - Then
    \[
    f(x) \approx F_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)
    \]
    - The coefficients are determined by
      \[
      \int_{-\pi}^{\pi} (f(x) - F_n(x))^2 \, dx = \min
      \]
      - We saw that the basis functions are orthogonal and that
        \[
        a_k = \int_{-\pi}^{\pi} f(x) \cos kx \, dx
        \]
        \[
        b_k = \int_{-\pi}^{\pi} f(x) \sin kx \, dx
        \]
- A significant drawback of Fourier series is that any change in an arbitrarily small subinterval
of $[-\pi, \pi]$ affects all the coefficients of the Fourier Series.

- This can be overcome in many ways, including modifications of the Fourier Series.

- However, the most effective way of handling this, with many additional useful properties, is based on wavelets.

- The best introduction, in my opinion, to wavelets, is still


- Daubechies is one of the primary pioneers and developers of wavelets. Check the wikipedia for more information on her many distinctions and accomplishments.

- We’ll just look at one example, the subject of the first of Daubechies’ lectures, which, however, will introduce most of the key ideas.

**The Haar Wavelet**

- Wavelets are translations and dilations of a mother wavelet. In the special case of the Haar wavelet, that function is given by

$$
\psi(x) = \begin{cases} 
1 & \text{if } 0 \leq x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} \leq x < 1 \\
0 & \text{else}
\end{cases}
$$

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introduced by Alfred Haar, 1885-1933, in 1909
• The mother wavelet plays essentially the same role as the sine (or cosine) function in Fourier series.

• Note that the Haar wavelet has local support.

**Translation and Dilation**

• For integers $m$ and $n$ we define a Haar wavelet to be the function

\[
\psi_{m,n}(x) = 2^{-\frac{m}{2}} \psi \left( 2^{-m} x - n \right).
\]

• Note that the support of $\psi_{m,n}$ is the interval $[2^m n, 2^m (n + 1)]$. The length of the support is $2^m$.

• A major property of Haar wavelets is that they are orthonormal, i.e.,

\[
\int_{\mathbb{R}} \psi_{m,n} \psi_{m',n'} = \begin{cases} 
1 & \text{if } m = m' \text{ and } n = n' \\
0 & \text{else}
\end{cases}
\]
• This is easy to see. We first observe that

\[ \int_{\mathbb{R}} \psi_{m,n}^2 = \int_{n^{2^m}}^{(n+1)^{2^m}} 2^{-m} dx = 2^{-m}2^m = 1. \]

Now assume that \( n \neq n' \) and \( m = m' \). Then the supports of \( \psi_{m',n'} \) and \( \psi_{m,n} \) do not overlap and

\[ \int_{\mathbb{R}} \psi_{m,n} \psi_{m',n'} = 0. \]

Finally, suppose that \( m \neq m' \), and without loss of generality suppose that \( m < m' \). Then the support of \( \psi_{m,n} \) lies entirely in a region where \( \psi_{m',n'} \) is constant. Since \( \int_{\mathbb{R}} \psi_{m,n} = 0 \) we get that

\[ \int_{\mathbb{R}} \psi_{m,n} \psi_{m',n'} = 0. \]

**Approximation of an \( L_2 \) function**

• Suppose \( f \) is a function defined on the real line and satisfying

\[ \int_{\mathbb{R}} f^2 < \infty \]

• We contemplate approximating \( f \) by a function \( g \) in the 2-norm:

\[ \|f\| = \sqrt{\int_{\mathbb{R}} f^2} \]
Any $L_2$ function $f$, i.e., any function $f$ that satisfies
\[ \int_{\mathbb{R}} f^2 < \infty \]
can be arbitrarily well approximated in the sense that for all $\epsilon > 0$ there exists a finite linear combination $A_\epsilon$ of Haar wavelets such that
\[ \int_{\mathbb{R}} (f - A_\epsilon)^2 < \epsilon \]

- This fact is highly counterintuitive (in my opinion) since the integral of any linear combination of the Haar wavelets is zero, whereas the integral of $f$ may not be zero.

- But let us press ahead.

- First observe that any $L_2$ function $f$ can be arbitrarily well approximated by a function $f_0$ which is constant on each interval $[\ell 2^{-J_0}, (\ell + 1)2^{-J_0}]$ and whose support is $[-2^{J_1}, 2^{J_1}]$ where $J_0$ and $J_1$ are sufficiently large.

- We now come to the key idea. We write
\[ f^0 = f^1 + \delta_1 \]

where $f^1$ is an approximation of $f^0$ that is piecewise constant on subintervals of twice the size of those on which $f^0$ is constant.
• the value of $f^1$ on one of the new, twice as large, intervals is obtained by averaging the values of the original two subintervals.

• Note that $\delta^1 = f^1 - f^0$ is a linear combination of Haar Wavelets!

\[ f^0 = f^1 + \sum_{\ell} c_{-J_0+1, \ell} \psi_{-J_0+1, \ell}. \]

• Naturally, we apply the same trick to $f^1$, etc.

• Eventually we get

\[ f^0 = f^{J_0+J_1} + \sum_{m=-J_0+1}^{J_1} \sum_{\ell} c_{m, \ell} \psi_{m, \ell}. \]

• $f^{J_0+J_1}$ consists of two constant pieces on the original support.

• $f^{J_0+J_1+1}$ consists of one constant piece on the original support.

• However, we continue, just widening the support.

• At every stage we average a constant with zero, halve the function value, and double the support.
• In due course we obtain

\[ f^0 = \sum_{m=-J_0+1}^{J_1+K} \sum_{\ell} c_{m,\ell} \psi_{m,\ell} = f^{J_0+J_1+K} \]

• So we have written our original function as a linear combination of Haar Wavelets + a remainder that is arbitrarily small.

• Notice that we have sets of wavelets for every length scale (corresponding to a fixed value of \( m \)).

• This is referred to as **multiresolution analysis**

• All basis functions have local support.

• A local phenomenon affects only the coefficients of those basis functions that are non-zero at the location of the phenomenon.

• We considered the Haar wavelet for simplicity. It’s not unreasonable because many technological phenomena are in fact discrete and piecewise constant.

• However, we can of course use different types of basis functions, particularly smoother ones. For examples, see Figure 1.8 on page 15 of Daubechies’ book. It is similar to Figure 2\(^{-2-}\).

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\( ^{-2-} \) from https://www.intechopen.com/source/html/ 49109/media/image8.png
Figure 2. Some Orthogonal Wavelets.