Review

• polynomial interpolant: $C^\infty$, global, error may not converge to zero.

• piecewise linear interpolant: only $C^0$, local, Error goes to zero like $O(h^2)$.

• piecewise cubic Hermite interpolant: $C^1$, local, Error goes to zero like $O(h^4)$, requires derivative values.

Cubic Splines

• This is another classic subject. Here are two classic references:


• A recent Matlab oriented text is:

• Again, assume we are given data \((x_i, y_i), i = 0, \ldots, n\) where the knots are ordered:

\[ a = x_0 < x_1 < \ldots < x_n = b \]

• The word “spline” derives from a mechanical gadget that used to be used to draw smooth curves.

• Pass an elastic wire through some points and let it adjust itself so that it minimizes its strain energy.
• The curvature of the curve \( y = s(x) \) is

\[
\kappa(x) = \left| \frac{s''}{(1 + s'^2)^{3/2}} \right|.
\]

• If \( s' \) is small (or close to constant) the curvature is approximately proportional to \( s'' \).

• So the mechanical idea can be approximated by the mathematical problem of finding the interpolating function \( s \) that interpolates, i.e.,

\[
s(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n
\]

and that minimizes the integral

\[
\int_a^b (s''(x))^2 \, dx = \min
\]

• It is a great exercise to show that the solution of this \textbf{variational problem} is a cubic spline, i.e., a function that is cubic on each subinterval

\[
I_i = [x_{i-1}, x_i]
\]

and that is twice differentiable on \([a, b]\).
Count parameters and conditions
Boundary Conditions

• Three kinds of boundary conditions are in frequent use:

1. **natural end conditions.** Let

   \[ s''(a) = s''(b) = 0 \]

   • This is called **natural** since the above discussed wire would be linear to the left of \( a \) and to the right of \( b \).

2. **Forced End Condition.** Let

   \[ s'(a) = A, \quad s'(b) = B. \]

   This means forcing the wire to point in certain directions at the endpoints. The obvious question is how to pick \( A \) and \( B \).

3. **Not-a-Knot** condition. Force the spline to be three times differentiable at \( x_1 \) and \( x_{n-1} \):

   \[
   \begin{align*}
   \lim_{x \to x_1^-} s'''(x) &= \lim_{x \to x_1^+} s'''(x) \\
   \lim_{x \to x_{n-1}^-} s'''(x) &= \lim_{x \to x_{n-1}^+} s'''(x)
   \end{align*}
   \]

   • A piecewise cubic function that is three times differentiable is actually cubic, so \( x_1 \) and \( x_{n-1} \) are not knots. The resulting spline is cubic, rather than piecewise cubic, on \([x_0, x_2]\) and \([x_{n-2}, x_n]\).
• A **Cardinal Spline** is a spline satisfying the cardinal condition

\[ s(x_i) = \delta_{ij} \]

The support of a cardinal spline is the entire interval \([a, b]\) (except at the knots). Geometrically, if you move any one data point the wire wiggles everywhere.

• This means that **Spline interpolation is global.**
B-splines

- **B-splines** are splines with support that is small as possible, which for cubic splines means 4 intervals.

- B-splines form a large subject.
Polynomial Precision

- We say that an interpolation scheme has polynomial precision $k$ if the interpolant of a polynomial of degree up to $k$ is that polynomial.

- We also say that the interpolation scheme reproduces polynomials of degree up to $k$.

- Polynomial interpolation of $n + 1$ data reproduces polynomials of degree up to $n$.

- Piecewise linear interpolation reproduces polynomials of degree up to 1.

- Piecewise Cubic Hermite interpolation reproduces polynomials of degree up to 3.

- What about cubic spline interpolation:
Computation of Cubic Splines

• We only have to solve a tridiagonal linear system!


• exercise: Verify everything and work out the details!

• The second derivative of a cubic spline $s$ is piecewise linear and continuous. Letting

$$M_i = s''(x_i) \quad \text{and} \quad I_i = [x_{i-1}, x_i]$$

we set

$$h_i = x_i - x_{i-1}$$

as before and get for $x \in I_i$:

$$s''(x) = M_{i-1} \frac{x_i - x}{h_i} + M_i \frac{x - x_{i-1}}{h_i}.$$

• Integrating $s''$ twice gives

$$s(x) = M_{i-1} \frac{(x_i - x)^3}{6h_i} + M_i \frac{(x - x_{i-1})^3}{6h_i} + c_i(x_{i-1}-x) + d_i(x-x_{i-1})$$

where the $c_i$ and $d_i$ are as yet undetermined integration constants.
We must have

\[ s(x_{i-1}) = y_{i-1} \quad \text{and} \quad s(x_i) = y_i . \]

This determines the constants \( c_i \) and \( d_i \) and gives (still for \( x \in I_i \))

\[
s(x) = M_{i-1} \frac{(x_i - x)^3}{6h_i} + M_i \frac{(x - x_{i-1})^3}{6h_i} \\
+ \left( y_{i-1} - \frac{M_{i-1} h_i^2}{6} \right) \frac{x_i - x}{h_i} \\
+ \left( y_i - \frac{M_i h_i^2}{6} \right) \frac{x - x_{i-1}}{h_i} \quad (1) \]

We still need to define the \( M_i \). They can be obtained from the \( C^1 \) conditions

\[
\lim_{x \to x_i^-} s'(x) = \lim_{x \to x_i^+} s'(x) .
\]

Differentiating in (1) gives:

\[
S'(x) = - M_{i-1} \frac{(x_i - x)^2}{2h_i} + M_i \frac{(x - x_{i-1})^2}{2h_i} \\
+ \frac{y_i - y_{i-1}}{h_i} + \frac{M_{i-1} - M_i}{6} h_i \quad (2)
\]

Evaluating (2) and the expression on the next interval at \( x_i \) and equating the two expression
gives the equation
\[
\frac{h_i}{2}M_i + \frac{M_{i-1} - M_i}{6}h_i + \frac{y_i - y_{i-1}}{h_i} =
\]
\[- M_i \frac{h_{i+1}}{2} + \frac{y_{i+1} - y_i}{h_{i+1}} + \frac{M_i - M_{i+1}}{6}h_{i+1}
\]
which can be rewritten as
\[a_i M_{i-1} + b_i M_i + c_i M_{i+1} = d_i\]
where
\[
a_i = \frac{h_i}{6}, \quad b_i = \frac{1}{3}(h_i + h_{i+1}), \quad c_i = \frac{h_{i+1}}{6}, \quad d_i = \frac{y_{i-1} - y_i}{h_i} + \frac{y_{i+1} - y_i}{h_{i+1}}
\]
\[i = 1, 2, \ldots, n - 1\]

These are \(n - 1\) equations in \(M_0, \ldots, M_n\).

- The two missing conditions are the boundary conditions. For example, natural splines have the boundary conditions
  \[M_0 = M_n = 0.\]
  Adding these as the first and last equations to (3) gives a square tridiagonal \((n+1) \times (n+1)\) tridiagonal linear system.

- Exercise: work out the missing equations for forced end and knot-a-knot conditions.