• Recall the **Lagrange Interpolation Problem**: Given \((x_i, y_i), i = 0, 1, 2, \ldots, n\), find the polynomial

\[
p(x) = \sum_{j=0}^{n} \alpha_j x^j
\]  

such that

\[
p(x_i) = y_i \quad \text{for all} \quad i = 0, 1, \ldots, n
\]

• The polynomial \(p\) written in the form (1) is said to be in its **power form** or **standard form**.

• We saw yesterday that the Lagrange Interpolation problem has a unique solution if and only if the data sites \(x_i\) are **distinct**, i.e.,

\[
i \neq j \implies x_i \neq x_j.
\]

• We saw that this is so by observing that the determinant of the coefficient matrix in the linear system (2) is non-zero if and only if (3) holds.

• Another way to see that this is true is by observing that the homogeneous problem where
\( y_i = 0 \) for all \( i \) only has the zero polynomial as its solution.

- You might think that the existence and uniqueness result is trivial or obvious. We have \( n + 1 \) equations in \( n + 1 \) unknowns, a square linear system, and such systems usually have a unique solution.
Here are a couple of cautionary examples.

- **Example 1:** Find a quadratic polynomial $q$ such that

\[ q(-1) = A, \quad q'(0) = B, \quad \text{and} \quad q(1) = C. \]
• **Example 2:** Given the 3 data

\[(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad \text{and} \quad (x_3, y_3, z_3)\]

find a linear function

\[L(x, y) = ax + by + c\]

such that

\[L(x_i, y_i) = z_i, \quad i = 1, 2, 3.\]
The Power Form

• We saw yesterday that we can solve the Lagrange Interpolation Problem by solving the linear system

\[ V_n a = y \]

where

\[
V_n = \begin{bmatrix}
1 & x_0 & x_0^2 & \ldots & x_0^{n-1} & x_0^n \\
1 & x_1 & x_1^2 & \ldots & x_1^{n-1} & x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_{n-1} & x_{n-1}^2 & \ldots & x_{n-1}^{n-1} & x_{n-1}^n \\
1 & x_n & x_n^2 & \ldots & x_n^{n-1} & x_n^n \\
\end{bmatrix}
\]

is the Vandermonde Matrix,

\[ a = [\alpha_0 \quad \alpha_1 \quad \ldots \quad \alpha_n]^\top \]

is the coefficient vector, and

\[ y = [y_0 \quad y_1 \quad \ldots \quad y_n]^\top \]

is the data vector or the right hand side.

• We also saw that

\[ |V_n| = \prod_{j>i}(x_j - x_i) \]

\[ = (x_1 - x_0)(x_2 - x_0) \ldots (x_n - x_{n-1}). \]

• Solving the Vandermonde system gives us the coefficients of the interpolant in its power form.

\[^\text{-1}^\text{– Alexandre-Théophile Vandermonde, 1735–1796.}\]
The Lagrange Form

• The Lagrange form of the interpolant is

\[ p(x) = \sum_{i=0}^{n} y_i L_i(x) \]  \hspace{1cm} (4)

where the \( L_i \) are suitable polynomials.

• The form (4) is a special case of the more general concept of using the data as coefficients multiplying suitable basis functions. In general we obtain the \textbf{cardinal form} of the interpolant this way.

• What does \textit{suitable} mean?

\[ L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} \]

\[ = \prod_{j \neq i} \frac{(x-x_j)}{(x_i-x_j)} \prod_{j=i} (x_i-x_j) \]
\[ n = 2 \quad (x_0, y_0), (x_1, y_1), (x_2, y_2) \]

\[ p(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 \]
The Newton Form

• With both the power form or the cardinal form we have to start from scratch if we add a new point to our data.

• It would be nice if we could get the new interpolant by modifying the old interpolant suitably. This gives rise to the **Newton Form** in this particular case.

\[
(x_0, y_0) \quad (x_1, y_1) \quad \cdots \quad (x_n, y_n)
\]

\[ P_n(x) \quad \text{interpolates at} \quad x_0, \ldots, x_n \]

\[ P_0(x) = y_0 \]

\[ P_1(x) = P_0(x) + g_1(x-x_0) = y_0 + \frac{y_1 - y_0}{x_1 - x_0} (x-x_0) \]

\[ \gamma_1 = \frac{y_1 - y_0}{x_1 - x_0} \]

\[ P_2(x) = P_1(x) + g_2(x-x_1)(x-x_0) \]

\[ P_2(x_2) = y_2 = P_1(x_2) + g_2(x_2-x_1)(x_2-x_0) \]
\[ g_2 = \frac{y_2 - p_n(x_n)}{(x_2 - x_1)(x_2 - x_0)} \]

\[ p_3(x) = p_2(x) + b_3(x - x_2)(x - x_1)(x - x_0) \]

\[ b_3 = \frac{y_3 - p_2(x_3)}{(x_3 - x_2)(x_3 - x_1)(x_3 - x_0)} \]

\[ p_n(x) = p_{n-1}(x) + \frac{y_n - p_{n-1}(x_n)}{(x_n - x_{n-1}) \cdots (x_n - x_0)} \]
Blending

- This is also an instance of a general idea. Suppose we have solutions of subproblems of our interpolation problem. How we can combine those partial solutions to obtain a solution of the whole problem? In general this process is called blending.

- In our special case suppose we have two polynomials:
  - $p_0(x)$ interpolates at $x_1, x_2, \ldots, x_n$, and
  - $p_n(x)$ interpolates at $x_0, x_1, \ldots, x_{n-1}$.

- How can we combine $p_0$ and $p_n$ to obtain $p$ which interpolates at all data sites, i.e.,

\[ p(x_i) = y_i, \quad i = 0, 1, \ldots, n \]

\[
p(x) = \frac{x-x_0}{x_n-x_0} p_0(x) + \frac{x-x_n}{x_0-x_n} p_n(x)
\]

\[0 \leq i \leq n\]

\[
p(x_i) = \frac{x_i-x_0}{x_n-x_0} p_0(x_i) + \frac{x_i-x_n}{x_0-x_n} p_n(x_i)
\]

\[
= y_i \left( \frac{x_i-x_0}{x_n-x_0} + \frac{x_i-x_n}{x_0-x_n} \right)
\]

\[
= y_i \frac{x_i-x_0-(x_i-x_n)}{x_n-x_0} = \frac{x_n-x_0}{x_n-x_0} y_i
\]

\[= y_i.\]
• A crucial point is that since the interpolant is unique all the approaches we discussed give the same polynomial!

• **Exercise:** It’s worthwhile to demonstrate this in a simple case.

**Summary of Key Ideas**

• **Vandermonde Matrix:** set up and solve a square linear system. Existence is equivalent to Uniqueness.

• **Lagrange Form**, also called **cardinal form:** use the data as coefficients and find suitable basis functions.

• **Newton Form** allows to add one data point at a time.

• **Blending:** combine partial solutions to obtain a complete solution.
Accuracy

• Suppose the data are values of a given function:

\[ y_i = f(x_i) \]

• What can we say about the error

\[ E(x) = f(x) - p(x) \]

for values of \( x \) that are not one of the data sites.

• That’s our topic for Monday. I will have notes for that case, but I want to present the result in a sort of brainstorming session. Thus I will put the notes online only after we meet on Monday.