• Recall the **Lagrange Interpolation Problem**: Given \((x_i, y_i), \ i = 0, 1, 2, \ldots, n\), find the polynomial

\[
p(x) = \sum_{j=0}^{m} \alpha_j x^j
\]  

such that

\[
p(x_i) = y_i \quad \text{for all} \quad i = 0, 1, \ldots, n
\]  

• The polynomial \(p\) written in the form (1) is said to be in its **power form** or **standard form**.

• We saw yesterday that the Lagrange Interpolation problem has a unique solution if and only if the data sites \(x_i\) are distinct, i.e.,

\[
i \neq j \quad \implies \quad x_i \neq x_j.
\]  

• We saw that this is so by observing that the determinant of the coefficient matrix in the linear system (2) is non-zero if and only if (3) holds.

• Another way to see that this is true is by observing that the homogeneous problem where
\( y_i = 0 \) for all \( i \) only has the zero polynomial as its solution.

- You might think that the existence and uniqueness result is trivial or obvious. We have \( n + 1 \) equations in \( n + 1 \) unknowns, a square linear system, and such systems usually have a unique solution.
Here are a couple of cautionary examples.

- **Example 1:** Find a quadratic polynomial $q$ such that

$$q(-1) = A, \quad q'(0) = B, \quad \text{and} \quad q(1) = C.$$ 

$$q(x) = ax^2 + bx + c$$
$$q'(x) = 2ax + b$$

$$
\begin{bmatrix}
1 & -1 & 1 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} =
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
$$

$B$ must be zero.
• **Example 2:** Given the 3 data \((x_1, y_1, z_1), (x_2, y_2, z_2),\) and \((x_3, y_3, z_3)\)

find a linear function

\[ L(x, y) = ax + by + c \]

such that

\[ L(x_i, y_i) = z_i, \quad i = 1, 2, 3. \]
The Power Form

- We saw yesterday that we can solve the Lagrange Interpolation Problem by solving the linear system

\[ V_n a = y \]

where

\[
V_n = \begin{bmatrix}
1 & x_0 & x_0^2 & \ldots & x_0^{n-1} & x_0^n \\
1 & x_1 & x_1^2 & \ldots & x_1^{n-1} & x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_{n-1} & x_{n-1}^2 & \ldots & x_{n-1}^{n-1} & x_{n-1}^n \\
1 & x_n & x_n^2 & \ldots & x_n^{n-1} & x_n^n
\end{bmatrix}
\]

is the Vandermonde matrix,

\[ a = [\alpha_0 \quad \alpha_1 \quad \ldots \quad \alpha_n] \]

is the coefficient vector, and

\[ y = [y_0 \quad y_1 \quad \ldots \quad y_n] \]

is the data vector or the right hand side.

- We also saw that

\[
|V_n| = \prod_{j>i}(x_j - x_i)
\]

\[
= (x_1 - x_0)(x_2 - x_0) \ldots (x_n - x_{n-1}).
\]

- Solving the Vandermonde system gives us the coefficients of the interpolant in its power form.

\[^{-1} \text{ Alexandre-Théophile Vandermonde, 1735–1796.}\]
The Lagrange Form

The Lagrange form of the interpolant is

\[ p(x) = \sum_{i=0}^{n} y_i L_i(x) \]

where the \( L_i \) are suitable polynomials.

The form (4) is a special case of the more general concept of using the data as coefficients multiplying suitable basis functions. In general we obtain the cardinal form of the interpolant this way.

What does suitable mean?

\[ L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij} \]

Kronecker Delta

\[ L_i(x) = \frac{(x-x_0)(x-x_i)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_i)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} \]

\[ = \prod_{j \neq i} (x-x_j) \prod_{j \neq i} (x_i-x_j) \]
\[ \rho(x) = \sum_{i=0}^{n} y_i \frac{\prod_{j \neq i} (x-x_j)}{\prod_{j \neq i} (x_i-x_j)} \cdot 0 \]

\[ \rho(x) = x^2 = \frac{(x-1)(x-2)}{(0-1)(0-2)} \cdot 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \cdot 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \cdot c \]

\[ = -x(x-2) + 2 \times (x-1) \]

\[ = x^2 \]
The Newton Form

- With both the power form or the cardinal form we have to start from scratch if we add a new point to our data.

- It would be nice if we could get the new interpolant by modifying the old interpolant suitably. This gives rise to the **Newton Form** in this particular case.

\[
\begin{align*}
    p_{n-1}(x_j) &= y_j, \quad j = 0, \ldots, n-1 \\
    (x_n, y_n) \\
    p_n(x) &= p_{n-1}(x) + \delta_n (x-x_0)(x-x_1) \ldots (x-x_{n-1}) \\
    p_n(x_n) &= y_n = p_{n-1}(x_n) + \delta_n \prod_{j \neq n} (x-x_j) \\
    \delta_n &= \frac{y_n - p_{n-1}(x_n)}{\prod_{j \neq n} (x_n - x_j)}
\end{align*}
\]
\[ P_0(x) = 0 \]
\[ g_1 = \frac{y_1 - P_0(x)}{y_1 - y_0} = \frac{1 - 0}{1 - 0} = 1 \]

\[ P_1(x) = 1 \cdot (x - 0) = x \]
\[ g_2 = \frac{y_2 - P_1(x_2)}{(x_2 - x_1)(x_2 - x_0)} = \frac{4 - 2}{(2-1)(2-0)} = 1 \]

\[ P_2(x) = P_1(x) + g_2 (x - x_0)(x - x_1) \]
\[ = x + (x - 0)(x - 1) \]
\[ = x + x^2 - x \]
\[ = x^2 \]
Blending

- This is also an instance of a general idea. Suppose we have solutions of subproblems of our interpolation problem. How we can combine those partial solutions to obtain a solution of the whole problem? In general this process is called blending.

- In our special case suppose we have two polynomials:
  - $p_0(x)$ interpolates at $x_1, x_2, \ldots, x_n$, and
  - $p_n(x)$ interpolates at $x_0, x_1, \ldots, x_{n-1}$.

- How can we combine $p_0$ and $p_n$ to obtain $p$ which interpolates at all data sites, i.e.,

$$p(x_i) = y_i, \quad i = 0, 1, \ldots, n$$

$$p(x) = \frac{x-x_0}{x_n-x_0} p_0(x) + \frac{x-x_n}{x_0-x_n} p_n(x)$$

$$i = 1, \ldots, n-1$$

$$p(x_i) = \frac{x_i-x_0}{x_n-x_0} y_i + \frac{x_i-x_n}{x_0-x_n} y_i$$

$$= y_i \left( \frac{x_i-x_0 - (x_i-x_n)}{x_n-x_0} \right) = y_i$$
• A crucial point is that since the interpolant is unique all the approaches we discussed give the same polynomial!

• **Exercise:** It’s worthwhile to demonstrate this in a simple case.

**Summary of Key Ideas**

• **Vandermonde Matrix:** set up and solve a square linear system. Existence is equivalent to Uniqueness.

• **Lagrange Form**, also called **cardinal form**: use the data as coefficients and find suitable basis functions.

• **Newton Form** allows to add one data point at a time.

• **Blending:** combine partial solutions to obtain a complete solution.
Accuracy

• Suppose the data are values of a given function:
  \[ y_i = p(x_i) \]

• What can we say about the error
  \[ E(x) = f(x) - p(x) \]

  for values of \( x \) that are not one of the data sites.

• That’s our topic for Monday. I will have notes for that case, but I want to present the result in a sort of brainstorming session. Thus I will put the notes online only after we meet on Monday.