Eigenvalue Problems

• all matrices $n \times n$ square, $x \in \mathbb{R}^n$, $\lambda$ is scalar (real, or possibly complex).

• $x \neq 0$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$ if

$$Ax = \lambda x$$  \hspace{1cm} (1)

• An eigenvector is determined only up to a non-zero factor:

$$A(kx) = kAx = k(\lambda x) = \lambda (kx).$$  \hspace{1cm} (2)

• Note that

$$Ax = \lambda x \iff Ax - \lambda x = 0$$

$$\iff (A - \lambda I)x = 0, \quad x \neq 0$$

$$\iff \det(A - \lambda I) = 0$$  \hspace{1cm} (3)

• The equation

$$\det(Ax - \lambda I) = 0$$  \hspace{1cm} (4)

is the characteristic equation of $A$. 

\[\text{Math 5610 Fall 2021} \quad \text{Notes of 10/6/21}\]
• The expression $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree $n$ with leading term $(-\lambda^n)$.

• why? \[ |A| = \sum_{\sigma} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \quad |A-\lambda I| \]

• $p(\lambda)$ is the **characteristic polynomial** of $A$.

• The eigenvalues are the roots of the characteristic polynomial. Thus $A$ has $n$ eigenvalues, properly counting multiplicity.

• Finding the eigenvalues by computing the characteristic polynomial and then finding the roots of that polynomial is a very ill-conditioned process, since the roots of a polynomial are very sensitive with respect to small changes in the coefficients.

• However, the opposite way works very well. Start with a polynomial, normalize it to have leading coefficient $(-1)^n$, giving a polynomial $p$, then construct a matrix, called the **companion matrix** of $p$, which has $p$ as its characteristic polynomial, and then compute the eigenvalues of $A$. See Problem 9 of hw 3 for specific formulas.

• The eigenvalues can be computed using standard software, most likely implementing the so-called **QR-Algorithm** (which is hugely complicated and mostly beyond our scope this semester).

• The matlab `roots` command find the roots of a polynomial that way.
Relationships

• Suppose we have \( n \) linearly independent eigenvectors \( x_i, i = 1, \ldots, n \) satisfying

\[
Ax_i = \lambda_i x_i, \quad i = 1, \ldots, n. \tag{5}
\]

Collect the eigenvalues and eigenvectors into matrices:

\[
X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \tag{6}
\]

• We can write (5) in matrix form as

\[
AX = X\Lambda \tag{7}
\]

which gives

\[
A = X\Lambda X^{-1} \quad \text{and} \quad \Lambda = X^{-1}AX. \tag{8}
\]

• You can use these formulas to construct a matrix with given eigenvalues and eigenvectors which is sometimes useful.

• The rows of \( X^{-1} \) are the **left eigenvectors** of \( A \):

\[
X^{-1}A = \Lambda X^{-1} \tag{9}
\]

• \( \Lambda = X^{-1}AX \) is an example of a **similarity transform** of \( A \).
• Two matrices \( A \) and \( B \) are \textbf{similar} if there exists a non-singular matrix \( T \) such that

\[
B = T^{-1}AT. \quad (10)
\]

• Similar matrices have the same eigenvalues:

\[
Ax = \lambda x \quad \Rightarrow \quad B(T^{-1}x) = T^{-1}ATT^{-1}x = T^{-1}Ax = \lambda(T^{-1}x) \quad (11)
\]

• The eigenvectors of \( B \) are \( T^{-1} \) times those of \( A \).

• If we have \( n \) linearly independent eigenvectors with real eigenvalues then we have a similarity transform to diagonal form.

• Of course the eigenvalues may be complex.

• However, the eigenvalues of a symmetric real matrix are real, see hw 3.
• Major issue: The eigenvectors of a matrix may not span the whole space of $\mathbb{R}^n$. If they don’t the underlying matrix is said to be defective.

• Examples of defective matrices:

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[y = 0\]

\[0x + 0 \cdot 0 = 0 \quad x \neq 0 \text{ arbitrary}\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}
\]

\[x + y = x \implies y = 0 \quad x \neq 0 \text{ arbitrary}\]
Consider a set of \( n \) linearly independent eigenvectors. Let \( X \) be the matrix that has these eigenvectors as its columns. Then

\[
X^{-1} X = I, \quad X = [x_1, \ldots, x_n], \quad Y = X^{-1} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix}
\]

and the left and right eigenvectors are biorthogonal:

\[
y_i^T x_j = 0 \quad \text{if} \quad i \neq j \quad (13)
\]

This can be useful for finding components corresponding to individual eigenvectors:

\[
A x_j = \lambda_j x_j, \quad \Rightarrow \quad y_i^T A x_j = \lambda_j y_i^T x_j = \lambda_j y_i^T y_i \quad (13)
\]

\[
\lambda_j y_i^T x_j = \lambda_i y_i^T x_j \quad (13)
\]

\[
\left( \lambda_i - \lambda_j \right) y_i^T x_j = 0 \quad \lambda_i \neq \lambda_j
\]

\[
y_i^T x_j = 0
\]
$X = \sum_{i=1}^{n} \alpha_i \cdot x_i$

$Z_i$, single evaulue

$Z_i \neq Z_j \quad j = 2, \ldots, n$

$y_i^T X = \sum_{i=1}^{n} \alpha_i y_i^T x_i$

$= \alpha_1 y_1^T x_1$

$\alpha_1 = \frac{y_1^T x}{y_1^T x_1}$
• Of course, it would be particularly nice to have an orthogonal set of eigenvectors. In that case we get

\[ AQ = Q\Lambda \quad Q^T = Q^{-1} \]

\[ \Lambda = Q^T AQ \]

\[ A = Q\Lambda Q^T \implies A \text{ is symmetric!} \quad (14) \]

• Only symmetric matrices have orthogonal sets of eigenvectors!

• The converse also holds: If \( A \) is symmetric it has an orthogonal set of eigenvectors. The proof is a bit complicated!
• Left and right eigenvectors corresponding to distinct eigenvalues are orthogonal:
Generically eigenvalues are distinct. (If they are not we can apply an arbitrarily small perturbation to the matrix that will make the eigenvalues distinct.)

But we may be in trouble if we have repeated eigenvalues.

As mentioned above, a matrix is defective if its eigenvectors do not span all of \( \mathbb{R}^n \). Recall that a matrix is singular if zero is one of its eigenvalues.

Singularity is unrelated to Defectiveness! Consider this table:

<table>
<thead>
<tr>
<th></th>
<th>singular</th>
<th>non-singular</th>
</tr>
</thead>
</table>
| defective     | \[
    \begin{bmatrix}
    0 & 1 \\
    0 & 0 \\
    \end{bmatrix}
\]
|               | \[
    \begin{bmatrix}
    1 & 1 \\
    0 & 1 \\
    \end{bmatrix}
\]
| non-defective | \[
    \begin{bmatrix}
    0 & 0 \\
    0 & 0 \\
    \end{bmatrix}
\]
|               | \[
    \begin{bmatrix}
    1 & 0 \\
    0 & 0 \\
    \end{bmatrix}
\]

Is there a degree of defectiveness? Such as there is a degree of singularity, i.e., ill-conditioning?

Yes! Discuss on Friday.
The Jordan Canonical Form

- Discuss if time permits.

- Two matrices $A$ and $B$ are similar if and only if they have the same **Jordan (Canonical) Form** (up to reordering the diagonal blocks).

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{bmatrix} \quad \text{where} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$  \hfill (16)

- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial. The **geometric multiplicity** of an eigenvalue is the number of linearly independent corresponding eigenvectors.

- Each Jordan block $J_i$ corresponds to one eigenvector.

- Thus the dimension of the space spanned by the eigenvectors of $J$ equals $k$.

- The Jordan Canonical Form is not computable in approximate arithmetic.
• Example:

\[
A = \begin{bmatrix}
1 & 2 & 2 & 2 & 3 & 1 & \cdots & 4 & 1 & 4 \\
2 & 1 & 2 & 3 & 1 & \cdots & 3 & 1 & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\end{bmatrix} = 0
\]

• A has 4 distinct eigenvalues
• They are: \{1, 2, 3, 4\}
• The geometric multiplicity of \(\lambda = 1\) is 1 and its algebraic multiplicity is 1.
• The geometric multiplicity of \(\lambda = 2\) is 3 and its algebraic multiplicity is 4.
• The geometric multiplicity of \(\lambda = 3\) is 1 and its algebraic multiplicity is 4.
• The geometric multiplicity of \(\lambda = 4\) is 1 and its algebraic multiplicity is 2.
• The dimension of the linear space spanned by the eigenvectors of \(A\) equals 6.