Eigenvalue Problems

• all matrices \( n \times n \) square, \( x \in \mathbb{R}^n \), \( \lambda \) is scalar (real, or possibly complex).

• \( x \neq 0 \) is an eigenvector of \( A \) with corresponding eigenvalue \( \lambda \) if
  \[ Ax = \lambda x \]

• An eigenvector is determined only up to a non-zero factor:
  \[ A(kx) = kAx = k(\lambda x) = \lambda(kx). \]

• Note that
  \[ Ax = \lambda x \iff Ax - \lambda x = 0 \]
  \[ \iff (A - \lambda I)x = 0, \quad x \neq 0 \]
  \[ \iff \det(A - \lambda I) = 0 \]

• The equation
  \[ \det(Ax - \lambda I) = 0 \]

  is the characteristic equation of \( A \).
• The expression \( p(\lambda) = \det(Ax - \lambda x) \) is a polynomial of degree \( n \) with leading term \((-\lambda^n)\).

• why?

\[
|A| = \sum_{C} \text{sign}(C) \prod_{j=1}^{n} a_{i,j} c_{j}
\]

• \( p(\lambda) \) is the characteristic polynomial of \( A \).

• The eigenvalues are the roots of the characteristic polynomial. Thus \( A \) has \( n \) eigenvalues, properly counting multiplicity.

• Finding the eigenvalues by computing the characteristic polynomial and then finding the roots of that polynomial is a very ill-conditioned process, since the roots of a polynomial are very sensitive with respect to small changes in the coefficients.

• However, the opposite way works very well. Start with a polynomial, normalize it to have leading coefficient \((-1)^n\), giving a polynomial \( p \), then construct a matrix, called the companion matrix of \( p \), which has \( p \) as its characteristic polynomial, and then compute the eigenvalues of \( A \). See Problem 9 of hw 3 for specific formulas.

• The eigenvalues can be computed using standard software, most likely implementing the so-called QR-Algorithm (which is hugely complicated and beyond our scope this semester).

• The matlab roots command find the roots of a polynomial that way.
**Relationships**

- Suppose we have \( n \) linearly independent eigenvectors \( x_i, \ i = 1, \ldots, n \) satisfying
  \[
  Ax_i = \lambda_i x_i, \quad i = 1, \ldots, n. \tag{1}
  \]

  Collect the eigenvalues and eigenvectors into matrices:

  \[
  X = [x_1 \ x_2 \ \ldots \ x_n] \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix}
  \]

- We can write (1) in matrix form as
  \[
  AX = X\Lambda \quad \iff \quad \Lambda X = X^{-1}AX.
  \]

- You can use these formulas to construct a matrix with given eigenvalues and eigenvectors which is sometimes useful.

- The rows of \( X^{-1} \) are the **left eigenvectors** of \( A \):
  \[
  X^{-1}A = \Lambda X \quad \iff \quad X^{-1} = \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix}
  \]

- \( \Lambda = X^{-1}AX \) is an example of a **similarity transform** of \( A \).
• Two matrices $A$ and $B$ are **similar** if there exists a non-singular matrix $T$ such that

$$B = T^{-1}AT.$$  

• Similar matrices have the same eigenvalues:

$$Ax = \lambda x \implies B(T^{-1}x) = T^{-1}ATT^{-1}x = T^{-1}Ax = \lambda T^{-1}x$$

• The eigenvectors of $B$ are $T^{-1}$ times those of $A$.

• If we have $n$ linearly independent eigenvectors with real eigenvalues then we have a similarity transform to diagonal form.

• Of course the eigenvalues may be complex.

• However, the eigenvalues of a symmetric real matrix are real, see hw 3.
• Major issue: The eigenvectors of a matrix may not span the whole space of $\mathbb{R}^n$. If they don’t the underlying matrix is said to be defective.

• Examples of defective matrices:

\[
A = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
x_2 = 0
\]

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

and non-zero multiples

\[
A = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
x_1 + x_2 \\
x_2
\end{bmatrix} = 1 \cdot \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[
x_1 + x_2 = x_1, \quad x_2 = 0
\]
Suppose we do have a set of $n$ linearly independent eigenvectors. Let $X$ be the matrix that has those eigenvectors as its columns. Then

$$X^{-1}X = I, \quad X = [x_1, \ldots, x_n], \quad Y = X^{-1} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix}$$

and the left and right eigenvectors are biorthogonal:

$$y_i^T x_j = 0 \quad \text{if} \quad i \neq j$$

This can be useful for finding components corresponding to individual eigenvectors:

$$X^{-1}x = \frac{1}{\sum_i \alpha_i x_i}$$

$$Z = \sum_{i=1}^n \alpha_i x_i$$

$$\|x_i\|_2 = \|y_i\|_2 = 1$$

$$Y_i^T Z = \sum_{i=1}^n \alpha_i y_i^T x_i$$

$$= \alpha_i y_i^T x_i$$

$$\alpha_i = \frac{y_i^T Z}{y_i^T x_i}$$
• Of course, it would be particularly nice to have an orthogonal set of eigenvectors. In that case we get

\[ AQ = Q\Lambda \quad \quad Q^T = Q^{-1} \]

\[ \Lambda = Q^T AQ \]

\[ A = Q\Lambda Q^T \implies A \text{ is symmetric!} \]

• Only symmetric matrices have orthogonal sets of eigenvectors!

• The converse also holds: If \( A \) is symmetric it has an orthogonal set of eigenvectors. The proof is a bit complicated!
• Left and right eigenvectors corresponding to distinct eigenvalues are orthogonal:

\[
\begin{align*}
Ax &= \lambda x \\
A^T x &= \mu x \\
\lambda^T Ax &= \lambda x^T \\
\lambda^T A^T x &= \mu x^T \\
\lambda^T x &= \mu x \\
(\lambda - \mu) x^T &= 0 \\
0 &= x^T
\end{align*}
\]
• Generically eigenvalues are distinct. (If they are not we can apply an arbitrarily small perturbation to the matrix that will make the eigenvalues distinct.)

• But we may be in trouble if we have repeated eigenvalues.

• As mentioned above, a matrix is **defective** if its eigenvectors do not span all of \( \mathbb{R}^n \). Recall that a matrix is **singular** if zero is one of its eigenvalues.

• Singularity is unrelated to Defectiveness! Consider this table:

<table>
<thead>
<tr>
<th></th>
<th>singular</th>
<th>non-singular</th>
</tr>
</thead>
</table>
| defective| \[
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\] |
| non-defective | \[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\] |

• Is there a degree of defectiveness? Such as there is a degree of singularity, i.e., ill-conditioning?

• Yes! Discuss tomorrow.
The Jordan Canonical Form

- Discuss if time permits.
- Two matrices $A$ and $B$ are similar if and only if they have the same *Jordan (Canonical) Form* (up to reordering the diagonal blocks).

\[
J = \begin{bmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_k
\end{bmatrix}
\]

where $J_i = \begin{bmatrix}
\lambda_i & 1 & 0 & \ldots & 0 \\
0 & \lambda_i & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & \lambda_i
\end{bmatrix}$

- The *algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic polynomial. The *geometric multiplicity* of an eigenvalue is the number of linearly independent corresponding eigenvectors.
- Each Jordan block $J_i$ corresponds to one eigenvector.
- Thus the dimension of the space spanned by the eigenvectors of $J$ equals $k$.
- The Jordan Canonical Form is not computable in approximate arithmetic.