Eigenvalue Problems

- all matrices $n \times n$ square, $x \in \mathbb{R}^n$, $\lambda$ is scalar (real, or possibly complex).

- $x \neq 0$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$ if

\[Ax = \lambda x\]  \hspace{1cm} (1)

- An eigenvector is determined only up to a non-zero factor:

\[A(kx) = kAx = k(\lambda x) = \lambda (kx).\]  \hspace{1cm} (2)

- Note that

\[Ax = \lambda x \iff Ax - \lambda x = 0\]
\[\iff (A - \lambda I)x = 0, \quad x \neq 0\]
\[\iff \det(A - \lambda I) = 0\]  \hspace{1cm} (3)

- The equation

\[\det(A - \lambda I) = 0\]  \hspace{1cm} (4)

is the characteristic equation of $A$. 
• The expression $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree $n$ with leading term $(-\lambda^n)$.

• why?

• $p(\lambda)$ is the characteristic polynomial of $A$.

• The eigenvalues are the roots of the characteristic polynomial. Thus $A$ has $n$ eigenvalues, properly counting multiplicity.

• Finding the eigenvalues by computing the characteristic polynomial and then finding the roots of that polynomial is a very ill-conditioned process, since the roots of a polynomial are very sensitive with respect to small changes in the coefficients.

• However, the opposite way works very well. Start with a polynomial, normalize it to have leading coefficient $(-1)^n$, giving a polynomial $p$, then construct a matrix, called the companion matrix of $p$, which has $p$ as its characteristic polynomial, and then compute the eigenvalues of $A$. See Problem 9 of hw 3 for specific formulas.

• The eigenvalues can be computed using standard software, most likely implementing the so-called $QR$-Algorithm (which is hugely complicated and mostly beyond our scope this semester).

• The matlab roots command find the roots of a polynomial that way.
Relationships

• Suppose we have \( n \) linearly independent eigenvectors \( x_i, i = 1, \ldots, n \) satisfying

\[
Ax_i = \lambda_i x_i, \quad i = 1, \ldots, n. \tag{5}
\]

Collect the eigenvalues and eigenvectors into matrices:

\[
X = [x_1 \ x_2 \ \ldots \ x_n] \quad \text{and} \quad \Lambda = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \ddots \\
0 & \lambda_n
\end{bmatrix} \tag{6}
\]

• We can write (5) in matrix form as

\[
AX = X\Lambda \tag{7}
\]

which gives

\[
A = X\Lambda X^{-1} \quad \text{and} \quad \Lambda = X^{-1}AX. \tag{8}
\]

• You can use these formulas to construct a matrix with given eigenvalues and eigenvectors which is sometimes useful.

• The rows of \( X^{-1} \) are the left eigenvectors of \( A \):

\[
X^{-1}A = \Lambda X^{-1} \tag{9}
\]

• \( \Lambda = X^{-1}AX \) is an example of a similarity transform of \( A \).
• Two matrices $A$ and $B$ are **similar** if there exists a non-singular matrix $T$ such that

\[ B = T^{-1}AT. \]  \hspace{1cm} (10)

• Similar matrices have the same eigenvalues:

\[
Ax = \lambda x \implies B(T^{-1}x) = T^{-1}ATT^{-1}x = T^{-1}Ax = \lambda(T^{-1}x) \]  \hspace{1cm} (11)

• The eigenvectors of $B$ are $T^{-1}$ times those of $A$.

• If we have $n$ linearly independent eigenvectors with real eigenvalues then we have a similarity transform to diagonal form.

• Of course the eigenvalues may be complex.

• However, the eigenvalues of a symmetric real matrix are real, see hw 3.
• Major issue: The eigenvectors of a matrix may not span the whole space of $\mathbb{R}^n$. If they don’t the underlying matrix is said to be defective.

• Examples of defective matrices:
• Suppose we do have a set of \( n \) linearly independent eigenvectors. Let \( X \) be the matrix that has those eigenvectors as its columns. Then

\[
X^{-1} X = I, \quad X = [x_1, \ldots, x_n], \quad Y = X^{-1} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix}
\]

and the left and right eigenvectors are biorthogonal:

\[
y_i^T x_j = 0 \quad \text{if} \quad i \neq j
\]

• This can be useful for finding components corresponding to individual eigenvectors:
• Of course, it would be particularly nice to have an orthogonal set of eigenvectors. In that case we get

\[ AQ = Q\Lambda \quad \quad Q^T = Q^{-1} \]

\[ \Lambda = Q^T AQ \]

\[ A = Q\Lambda Q^T \implies A \text{ is symmetric!} \]

\[ (14) \]

• Only symmetric matrices have orthogonal sets of eigenvectors!

• The converse also holds: If \( A \) is symmetric it has an orthogonal set of eigenvectors. The proof is a bit complicated!
• Left and right eigenvectors corresponding to distinct eigenvalues are orthogonal:
• Generically eigenvalues are distinct. (If they are not we can apply an arbitrarily small perturbation to the matrix that will make the eigenvalues distinct.)

• But we may be in trouble if we have repeated eigenvalues.

• As mentioned above, a matrix is **defective** if its eigenvectors do not span all of \( \mathbb{R}^n \). Recall that a matrix is **singular** if zero is one of its eigenvalues.

• Singularity is unrelated to Defectiveness! Consider this table:

<table>
<thead>
<tr>
<th></th>
<th>singular</th>
<th>non-singular</th>
</tr>
</thead>
<tbody>
<tr>
<td>defective</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
</tbody>
</table>

|          | [ ]      | [ ]          |
| non-defective |          |              |

(15)

• Is there a degree of defectiveness? Such as there is a degree of singularity, i.e., ill-conditioning?

• Yes! Discuss on Friday.
The Jordan Canonical Form

- Discuss if time permits.

- Two matrices $A$ and $B$ are similar if and only if they have the same Jordan (Canonical) Form (up to reordering the diagonal blocks).

\[ J = \begin{bmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_k \end{bmatrix} \quad \text{where} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_i \end{bmatrix} \quad (16) \]

- The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial. The geometric multiplicity of an eigenvalue is the number of linearly independent corresponding eigenvectors.

- Each Jordan block $J_i$ corresponds to one eigenvector.

- Thus the dimension of the space spanned by the eigenvectors of $J$ equals $k$.

\[ \text{The Jordan Canonical Form is not computable in approximate arithmetic.} \]
Example:

\[
A = \begin{bmatrix}
1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 \\
& 1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 \\
& & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 \\
& & & 2 & 3 & 3 & 3 & 3 & 4 & 4 \\
& & & & 2 & 3 & 3 & 3 & 4 & 4 \\
& & & & & 2 & 3 & 3 & 4 & 4 \\
& & & & & & 2 & 3 & 4 & 4 \\
& & & & & & & 2 & 4 & 4 \\
& & & & & & & & 4 & 4 \\
& & & & & & & & & 4
\end{bmatrix}
\]

\[= 0 \quad (17)\]

- \(A\) has distinct eigenvalues
- They are:
  - The geometric multiplicity of \(\lambda = 1\) is and its algebraic multiplicity is .
  - The geometric multiplicity of \(\lambda = 2\) is and its algebraic multiplicity is .
  - The geometric multiplicity of \(\lambda = 3\) is and its algebraic multiplicity is .
  - The geometric multiplicity of \(\lambda = 4\) is and its algebraic multiplicity is .
  - The dimension of the linear space spanned by the eigenvectors of \(A\) equals .