Recall Newton’s Method for Systems:

Suppose $F$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

and we want to solve

$$F(x) = 0$$  \hspace{1cm} (1)

Here

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

and

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_n(x) \end{bmatrix},$$

Also let

$$\nabla F = \left[ \frac{\partial F_i}{\partial x_j} \right]_{i,j=1,2,...,n}$$

is the Jacobian of $F$. 

$$F(x) = A x - b$$
Newton’s Method is given by

\[ x^{(0)} \text{ given} \]

For \( k = 0, 1, 2, \ldots \) do:
1. Compute \( F(x^{(k)}) \) and \( A = \nabla F(x^{(k)}) \)
2. Solve

\[ As = -F(x^{(k)}) \]

(The vector \( s \) is called the Newton Step).
3. Let

\[ x^{(k+1)} = x^{(k)} + s \]

4. Repeat until

\[ \|s\| < \epsilon \]

(We stop when the length of the Newton step is less than some tolerance \( \epsilon \) that depends on the problem. For example, if we compute cartesian coordinates in the term project, \( \epsilon = 10^{-2} \), we want an accuracy of 1cm.)

Newton’s Method is often written as

\[ x^{(0)} \text{ given} \quad x^{(k+1)} = x^{(k)} - \left( \nabla F(x^{(k)}) \right)^{-1} F(x^{(k)}) \]

where \( k = 0, 1, 2, \ldots \). However, there is no need to compute the inverse of the Jacobian.

\[ 12.10: \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

\[ f(x) = 0 \]
• Example:

\[
F_1(x, y) = x^2 + y^2 - 1 = 0 \\
F_2(x, y) = y - x^4 = 0, \quad x_0 = y_0 = 1. 
\]

• slight change in notation ...

\[
\frac{\partial F_1}{\partial x} = 2x \\
\frac{\partial F_1}{\partial y} = 2y \\
\frac{\partial F_2}{\partial x} = -4x^3 \\
\frac{\partial F_2}{\partial y} = 1 \\
J(1, 1) = \begin{bmatrix} 2 & 2 \\ -4 & 1 \end{bmatrix} \\
F(1, 1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
-4x + y > 0 \\
y = 4x \\
2x + 8x = -1 \\
x = -1/10 \\
y = -9/10 \\
S = \begin{bmatrix} -9/10 \\ -4/10 \end{bmatrix} \\
x^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -9/10 \\ -4/10 \end{bmatrix} = \begin{bmatrix} 9/10 \\ 6/10 \end{bmatrix}
\]
For the receiver program in the term project we have equations like
\[ z = \|x_V - x_S\| - c(t_V - t_S) = 0 \]
where
- \(x_V\) is the unknown position of the vehicle
- \(x_S\) is the known position of the satellite
- \(t_V\) is the unknown time at which the vehicle receives the signals, and
- \(t_S\) is the known time at which the satellite broadcasts.

Let’s say
\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix}
\]

Then
\[
z = \left( \sum_{i=1}^{3} (\xi_i - \sigma_i)^2 \right)^{\frac{1}{2}} - c(t_V - t_S),
\]
and, for example,
\[
\frac{\partial z}{\partial \xi_i} = \frac{2(\xi_i - \sigma_i)}{2 \left( \sum_{i=1}^{3} (\xi_i - \sigma_i)^2 \right)^{\frac{1}{2}}} = \frac{(\xi_i - \sigma_i)}{\left( \sum_{i=1}^{3} (\xi_i - \sigma_i)^2 \right)^{\frac{1}{2}}}.
\]
• We already discussed this briefly, but it is worth reiterating: \( t_V \) and \( x_V \) have very different accuracy requirements. We want to know \( x_V \) within one centimeter and \( t_V \) within one centimeter divided by the speed of light. This

• This is bound to lead to numerical trouble.

• However, \( t_V \) enters our equations linearly, and can be eliminated!

• We simply take differences. For example, suppose we have data from two satellites, \( S_1 \) and \( S_2 \):

\[
\begin{align*}
z_1 &= \|x_V - x_{S_1}\| - c(t_V - t_{S_1}) &= 0 \\
z_2 &= \|x_V - x_{S_2}\| - c(t_V - t_{S_2}) &= 0
\end{align*}
\]

• Subtracting the second equation from the first gives the equation

\[
\|x_V - x_{S_1}\| - \|x_V - x_{S_2}\| + c(t_{S_1} - t_{S_2}) = 0
\]

which no longer contains the variable \( t_V \).

• So, for example, if we have data from four satellites, \( S_1, S_2, S_3, \) and \( S_4 \), say, we can form 3 such equations corresponding to

\[
S_1 - S_2, \quad S_1 - S_3, \quad \text{and} \quad S_1 - S_4
\]

and solve the resulting system of 3 equations in 3 unknowns.

• But we have data from more than four satellites, and we want to (and should) use all!
• So we get an **overdetermined** system.
• We have more equations than unknowns.
• To introduce the relevant idea, **discrete non-linear Least Squares**, consider and example:

\[
\begin{align*}
F_1(x, y) &= x + y - 2 = 0 \\
F_2(x, y) &= x^2 + y^2 - 2 = 0 \\
F_3(x, y) &= xy - 2 = 0
\end{align*}
\]

• We have 3 equations in 2 unknowns, there is no solution (check it out!).
• However, if there was in fact a solution we’d have

\[
f(x, y) = \sum_{i=1}^{3} F_i^2(x, y) = F_1^2(x, y) + F_2^2(x, y) + F_3^2(x, y) = 0.
\]

• Since we can’t have that we do the next best thing: Find \(x\) and \(y\) so as to minimize \(f\):

\[
f(x, y) = \sum F_i^2(x, y) = \min.
\]

• This idea generalizes in an obvious way to a system of \(m\) equations in \(n\) variables, where \(m > n\):

• Suppose we have a function

\[
F : \mathbb{R}^n \longrightarrow \mathbb{R}^m
\]
where $m > n$. Instead of solving the root finding problem

$$F(x) = 0$$

which has no solution we solve the nonlinear Least Squares problem

$$f(x) = \|F(x)\|^2 = (F(x))^T F(x) = \min \quad (2)$$

• $F$ is called the **residual** in this context. Instead of making the residual zero we make it as small as possible.

• To find a solution of (2) we find a stationary point where

$$\nabla f(x) = 0.$$  

• We can and should apply these ideas to the term project. Suppose we have data from $m + 1$ satellites $S_0, \ldots, S_m$. Then we find $x_V$ so as to minimize

$$f(x_V) = \sum_{i=1}^{m} (\|x_V - x_{S_0}\| - \|x_V - x_{S_i}\| + c(t_{S_0} - t_{S_i}))^2 = \min.$$  

• To solve this problem we need to solve the nonlinear $3 \times 3$ system

$$\nabla f = 0.$$  

• To solve the nonlinear system use Newton’s Method.
• This will give rise to a sequence of $3 \times 3$ positive definite linear systems.

• HW 1 asked you to describe Newton’s Method. This means you need to give explicit formulas for the relevant derivatives!
• Let’s look at this from a slightly more general point of view, with an important conclusion at the end.

• Suppose

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{where} \quad m > n.$$  

• We want $F(x) = 0$ but the system is overdetermined and so we settle for solving

$$f(x) = \|F(x)\|^2 = F(x)^T F(x) = \min.$$  

• To find the minimizer we solve the nonlinear system

$$\nabla f(x) = 2\nabla F(x)^T F(x) = 0$$

where $\nabla f$ is the gradient of $f$ and the Jacobian $\nabla F(x)$ is given by

$$B = \nabla F(x) = \left[ \frac{\partial F_i}{\partial x_j} \right]_{i=1,\ldots,m \atop j=1,\ldots,n}.$$  

• To solve the nonlinear system we need the Jacobian of $\nabla f$. That’s the Hessian $H(x)$ (the symmetric matrix of second order partial derivatives) of $f$.

• In our case we get

$$H(x) = \nabla (\nabla f) = \nabla^2 f = \nabla (2\nabla F^T F')$$

$$= 2 \left( (\nabla^2 F)^T F + \nabla F^T \nabla F \right)$$

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Here, as before, \( B = \nabla F \) is an \( m \times n \) matrix. \( \nabla^2 F \) is an \( m \times n \times n \) tensor. It consists of \( n \times n \) layers, each of which is the matrix of second order partial derivatives of a component of \( F \).

However, the residual \( F \) is very small in our project!

- So we can safely ignore the tensor term and approximate

\[
H(x) \approx 2\nabla F^T \nabla F = B^T B
\]

- It’s easy to see that \( B^T B \) is positive definite.

- This concludes our discussion of the solution of nonlinear equations.