Variational Principles

and their Euler Equations


• Nature likes to minimize things. A minimization problem whose solution is a function is called a variational principle.

• Many DEs arise from variational principles.

• Suppose

\[ u = u(x), \quad u(0) = A, \quad \text{and} \quad u(1) = B. \]

We want to find \( u \) such that for some given function \( F \)

\[ I(u) = \int_0^1 F(x, u(x), u'(x)) \, dx = \min \quad (1) \]

• The space from which \( u \) is to be chosen matters a great deal. Essentially it is the set of all functions for which \( I(u) \) is defined and finite. However, we will focus on other issues. See Strang and Fix for more info.

• An important special case of (1) is the Sturm-Liouville Problem

\[ F(x, u, u') = p(x)(u'(x))^2 + q(x)u^2(x) - 2f(x)u(x) \quad (2) \]
which describes, for example, the deflection of a loaded beam, or the temperature in a rod.

- The Sturm Liouville Problem (2) has a unique solution if
  
  \[ p(x) \geq \epsilon > 0 \quad \text{and} \quad q(x) \geq 0. \]

**Obtaining the Euler Equation**

- Leonard Euler, 1707–1783.
- How does (1) lead to a differential equation?
- Introduce a **variation** \( \Phi \) with
  
  \[ \Phi(0) = \Phi(1) = 0 \]

  and consider the scalar valued function of a single variable
  
  \[ s(\epsilon) = I(u + \epsilon \Phi). \]

- Since \( \Phi \) does not upset the boundary conditions, if \( u \) minimizes \( I(u) \) then \( \epsilon = 0 \) minimize \( s(\epsilon) \). Thus we must have
  
  \[ s'(0) = 0. \]

- Differentiating with respect to \( \epsilon \) in
  
  \[ s(\epsilon) = \int_0^1 F(x, u + \epsilon \Phi, u' + \epsilon \Phi')dx \]

  gives
  
  \[ s'(\epsilon) = \int_0^1 F_u \Phi + F_{u'} \Phi' dx, \quad s'(0) = 0. \]

- The key idea now is to get an integral of a product, one factor of which is \( \Phi \). Since \( \Phi \) is arbitrary (except at the boundary) this would imply that the other factor is zero.
• To get there we use **integration by parts**:

\[
\int_0^1 u'v = uv \bigg|_0^1 - \int_0^1 uv'
\]

• We obtain

\[
s'(\epsilon) = \int_0^1 F_u \Phi + F_u' \Phi'\,dx
\]

\[
= \int_0^1 F_u \Phi\,dx + \Phi F_u' \bigg|_0^1 - \int_0^1 \Phi \frac{d}{dx} F_u'\,dx
\]

\[
= 0
\]

\[
\Phi(F_u - \frac{d}{dx} F_u')\,dx
\]

\[
= 0 \quad \text{for all } \Phi
\]

• This implies that

\[
F_u - \frac{d}{dx} F_u' = 0
\]

which is the **Euler Equation** of (1).

• In the special case of the Sturm-Liouville Problem (2) we have

\[
F = p(u')^2 + qu^2 - 2fu
\]

and the Euler equation becomes

\[
F_u - \frac{d}{dx} F_u' = -2f + 2qu - 2\frac{d}{dx} pu' = 0.
\]

• In the even more special case that \(p\) is constant this turns into

\[
qu - pu'' = f
\]
• In general,
\[
\frac{d}{dx} F_u' = F_{xu'} + F_{uu'}u' + F_{u'u''u''}
\]

• We started with the Variational Principle and came up with the Euler Equation.

• Textbooks on Finite Elements, including Strang and Fix, usually go the other way:

• Suppose we want to solve the Sturm Liouville Problem

\[
Lu = -\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u = f(x), \quad u(0) = u(1) = 0.
\]

• The linear differential equation (3) is related to the variational principle

\[
I(v) = (Lv, v) - 2(f, v)
\]

where

\[
(f, g) = \int_0^1 f(x)g(x)dx.
\]

**The Ritz Method**

• Let’s return to the Sturm Liouville Problem

\[
I(u) = \int_0^1 p(x)(u'(x))^2 + q(x)u^2(x) - 2f(x)u(x) = \min
\]

subject to the boundary conditions

\[
u(0) = A \quad \text{and} \quad u(1) = B
\]

• We want to minimize \( I \) over the affine space of all functions for which \( I(u) \) is well defined, and which satisfy the boundary conditions.
• We could do this by solving the Euler Equation.
• But can we deal with (6) directly?
• The basic idea of the Ritz Method is to minimize (6) over a finite dimensional subspace!
• That space could contain, for example, polynomials, eigenfunctions of $L$, or, most frequently, piecewise polynomial functions.
• The last choice gives rise to the finite element method.
• Note that the Euler Equation involves higher order derivatives than the variational principle.
• We impose boundary conditions by choosing the approximating subspace suitably. What if we don’t enforce boundary conditions?
• Recall
  \[ s'(\epsilon) = \int_0^1 \Phi \left( F_u - \frac{d}{dx} F_u' \right) dx + \Phi F_u' \bigg|_0^1 = 0. \]
• If $\Phi$ is arbitrary we must have that $F_u'$ is zero at 0 and 1.
• For our Sturm-Liouville Problems we have
  \[ F_u' = 2pu' \]
  which leads to the natural boundary condition
  \[ u'(0) = u'(1) = 0. \]
• By contrast, the Dirichlet conditions
  \[ u(0) = A \quad \text{and} \quad u(1) = B \]
  are called essential boundary conditions.
• Essential boundary conditions are built into the approximation subspace, and natural boundary conditions require no special subspace.
• There is a great deal more, e.g.,
  – Use approximating spaces that are piecewise polynomial and have small local supports. This gives rise to sparse linear systems.
  – What to do if we have a DE but no variational principle. The you use variational type methods (Collocation, Galerkin).
  – A truly huge area is doing all this for partial differential equations.

\[
(Lu)(x_i) = f(x_i) \quad Lu = f \\
\langle \psi_i, Lu \rangle = \langle \psi_i, f \rangle
\]

\[
(f, g) = \int f(x) g(x) dx = 0
\]

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \quad \Longleftrightarrow \quad \iint_{\Omega} u_{xx}^2 + u_{yy}^2 \, dx dy = \min
\]
PDEs

\[ x, y, z \quad \text{space} \]
\[ t \quad \text{time} \]

**Curse of Dimensionality**

\[ u_t = f(x, t, u, u_x, u_{xx}) \]
\[ u = u(x, t) \]
\[ 0 \leq x \leq 1 \]

**MOL Method of Lines**

\[ u_N(t) \approx u(x_N, t) \]
\[ u_N' = f(x_{N+1}, u_N, \frac{u_{N+1} - u_{N-1}}{2h}, \frac{u_{N+1} - 2u_N + u_{N-1}}{h^2}) \]