ODE-IVPs

- We consider the system of ordinary differential equations
  \[ y' = f(x, y) \]  
  with the initial condition
  \[ y(a) = y_0 \]
  where
  \[ a \leq x \leq b \]
  and \( y \) and \( f \) are in \( \mathbb{R}^m \).

- Thus we have a first order system of \( m \) ordinary differential equations.

- There is no loss of generality in assuming that our system is of first order. We can always convert a higher order system to a first order one by introducing more independent variables. For example, the second order problem
  \[ y'' = f(x, y, y') \]
  of \( m \) equations can be turned into the first order system
  \[ y' = w \]
  \[ w' = f(x, y, w) \]
  of \( 2m \) equations.

- The differential equation (1) is said to be autonomous if \( f \) does not depend on \( x \), i.e.,
  \[ y' = f(y). \]

- It is sometimes useful to assume that the underlying differential equation is autonomous. One can always convert a
DE into an autonomous system by considering $x$ a dependent variable (depending on a variable $t$ say) and defining it by the initial value problems

$$x' = 1, \quad x(a) = a.$$ 

We will assume that the initial value problem (1) and (2) has a unique solution. This is a subtle issue, but we will not pursue it in any depth.
Euler’s Method

• Consider again the initial value problem

\[ y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b \]

• We discretize the problem.

\[ h = \frac{b - a}{N}, \quad x_n = a + nh, \quad y_n \approx y(x_n), \quad f_n = f(x_n, y_n). \]

• Euler’s Method defines the approximation \( y_n \) as

\[ y_{n+1} = y_n + hf_n, \quad n = 0, 1, 2, \ldots \]

• Euler’s Method can be visualized graphically:
Error Propagation

• There are two sources of error in Euler’s Method:

• The error due to leaving the graph of some exact solution by going along the tangent. This is called the local truncation error.

• The error due to no longer being on the true solution in the first place.

• The local truncation error for Euler’s method is

\[ LTE = y(x_{n+1}) - y_{n+1} \]

assuming \( y_n = y(x_n) \)

\[ = y(x_{n+1}) - \left( y(x_n) + hf(x_n, y(x_n)) \right) \]

\[ = y(x_{n+1}) - y(x_n) - hy'(x_n) \]

\[ = -\frac{h^2}{2}y''(x_n) + \text{HOT}. \]

• Here, HOT stands for higher order terms. As we discussed in the past, they are the more negligible the smaller \( h \). As usual, we ignore them.

• The global truncation error for Euler’s or any other method is

\[ GTE = e_n = y(x_n) - y_n \]

• We ignore round-off errors.

Global Truncation Error for Euler’s Method

• Let’s look at the simple problem

\[ y' = y, \quad y(0) = 1, \quad y(x) = e^x. \]
Euler’s Method becomes

\[ y_{n+1} = y_n + hy_n = (1 + h)y_n \quad \implies \quad y_n = (1 + h)^n \]

Clearly

\[ e_n = y(x_n) - y_n = e^{nh} - (1 + h)^n. \]

Consider now a fixed number \( x \). We want to understand what happens to the error as \( h \) goes to zero, and the number of steps goes to infinity such that the product

\[ nh = x \]

remains constant. This concept is known as a **fixed station limit**.

We get

\[ e_n = e^x - (1 + h)^n = e^x - (1 + h)^{x/h} \]

It’s a simple Calculus exercise to verify that

\[
\lim_{h \to 0} (1 + h)^{x/h} = e^x \quad \text{and} \quad \lim_{h \to 0} \frac{d}{dh}(1 + h)^{x/h} = -\frac{xe^x}{2}
\]

Thus

\[ e_n = \frac{xe^x}{2}h + \text{HOT}. \]

The global truncation error is of order one less than the local truncation error.
HW 4 Comments

- Home work 4 asks you to compare results for four methods. I’m listing them here for reference. In all cases, let

\[ x_n = a + n \cdot h, \quad y_n \approx y(x_n) \text{ and } f_n = f(x_n, y_n) \]

where \( h \) is a given step size. You want to compute the sequence

\[ y_0, y_1, y_2, \ldots \]

Euler’s Method

We discussed that method above. It is given by

\[ y_{n+1} = y_n + hf_n. \]

The Trapezoidal Rule

\[ y_{n+1} = y_n + \frac{h}{2} (f_n + f_{n+1}). \]

- This method is implicit, \( y_{n+1} \) occurs as an argument of \( f_{n+1} \), and you need to solve an equation to compute. (Of course, that equation in problem 8 of hw 4 is linear.)

Simpson’s Rule

\[ y_{n+2} = y_n + \frac{h}{3} (f_n + 4f_{n+1} + f_{n+2}). \]

- This method is also implicit. In addition, it requires an additional starting value \( y_1 \) to get it going. As stated in the hw problem, use

\[ y_1 = y(x_1). \]
The standard Runge-Kutta Method

That method is defined in the actual hw problem, but for completeness, it’s given by

\[ y_{n+1} = y_n + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4) \]

where

\[ K_1 = f(x_n, y_n) \]
\[ K_2 = f \left( x_n + \frac{h}{2}, y_n + \frac{h}{2} K_1 \right) \]
\[ K_3 = f \left( x_n + \frac{h}{2}, y_n + \frac{h}{2} K_2 \right) \]
\[ K_4 = f \left( x_n + h, y_n + h K_3 \right) \].
Fundamentals of Stability

A simple Model of Error Propagation.

Suppose we have a stepwise procedure where the global (i.e., accumulated) error at each step is denoted by $e_n$. We assume that the error propagation is defined by

$$e_0 = 0 \quad \text{and} \quad e_{n+1} = \gamma e_n + \varepsilon, \quad n = 0, 1, 2, \ldots. \quad (4)$$

Here, $\varepsilon$ is the local error, $e_n$ is the global error (after $n$ steps), and $\gamma$ is the amplification factor. We also refer to $e_n$ simply as the error.

Note. Major simplifying assumptions are: The local error and the amplification factor are constant, and there is only one component of the error propagating in this fashion.

Note. We will encounter several variants of local and global errors.

Example, Quadrature. In quadrature the local errors simply add up, and so the amplification factor is 1.

Example, Euler’s Method. Suppose we solve the initial value problem

$$y' = y, \quad y(0) = 1 \quad (5)$$

which obviously has the solution $y(x) = e^x$. If we apply Euler’s method we obtain

$$y_{n+1} = (1 + h)y_n \quad \text{and} \quad e_n = e^{x_n} - y_n. \quad (6)$$
A simple manipulation shows that

\[ e_{n+1} = (1 + h)e_n + (e^h - (1 + h))e^x_n \]

\[ = (1 + h)e_n + e^x_n \sum_{i=2}^{\infty} \frac{h^i}{i!}. \]  

(7)

Thus the error propagation in this case is of the form (1), except that the local error is slowly growing with \( n \).

It is straightforward to verify that

\[ e_n = (1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1}) \varepsilon = \begin{cases} n\varepsilon & \text{if } \gamma = 1 \\ \frac{1-\gamma^n}{1-\gamma} \varepsilon & \text{else} \end{cases} \]

(8)

We ask what happens to the error as \( n \) tends to infinity. There are four cases to consider:

### \(|\gamma| < 1\)

In the limit, the global error is proportional to the local error:

\[ \lim_{n \to \infty} e_n = \frac{1}{1-\gamma} \varepsilon. \]  

(9)

### \(\gamma = 1\)

The error grows indefinitely but slowly:

\[ e_n = n\varepsilon. \]  

(10)

For example: Note that \( y_n = (1 + h)^n \). We get

\[ (1 + h)e_n + (e^h - (1 + h))e^x_n = (1 + h)(e^x_n - (1 + h)^n) + (e^h - (1 + h))e^x_n \]

\[ = e^x_n + he^x_n - (1 + h)^{n+1} + e^h e^x_n - e^x_n - he^x_n \]

\[ = e^{x_{n+1}} - (1 + h)^{n+1} \]

\[ = e_{n+1}. \]
The error alternates (between 0 and $\varepsilon$). This case is of no great practical interest.

The error is dominated by the exponentially growing term $\gamma^{n-1}\varepsilon$.

We now make our assumptions more realistic, contemplating a situation where we take increasingly many smaller and smaller steps to cover a given distance. As the step size decreases, the local error decreases also. Formally, our assumptions are

$$\varepsilon = h^{p+1}, \quad h = \frac{1}{n},$$

and we ask about the limits as

$$h \to 0, \quad n \to \infty.$$  (12)

**Note.** The number $p$ is usually an integer, and called the order of the method.

**Note.** More realistically, the amplification factor also depends on $h$ (and changes from step to step). There usually will be several superimposed process like the one we consider. We could also introduce a constant factor multiplying the local error. This would add no insights, however.

**Example.** In the above example, the order of Euler’s method is 1, and the local error is $-\frac{h^2}{2} y_n +$ higher order terms.

Substituting in (5) we obtain

$$e(h) = e_{1/h} = \begin{cases} 
\frac{1}{h} h^{p+1} = h^p & \text{if } \gamma = 1 \\
\frac{1 - \gamma^{1/h}}{1 - \gamma} h^{p+1} & \text{else} 
\end{cases}$$

(13)
We ask what happens as \( h \) tends to zero, and consider the same four cases as before.

\( |\gamma| < 1 \)

Clearly

\[
e(h) = O\left(h^{p+1}\right).
\]

(15)

In this cases, the global error is of the same order as the local error.

\( \gamma = 1 \)

We obtain

\[
e(h) = \left(\frac{1}{h}\right)h^{p+1} = h^p.
\]

(16)

The global error is of order one less than the local error. This is precisely the situation we encountered with quadrature.

\( \gamma = -1 \)

In this cases \( e_n \) oscillates between zero and \( h^{p+1} \). The global error is of the same order as the local error. However, this case is of no great practical interest.

\( |\gamma| > 1 \)

In this cases \( e(h) \) grows unboundedly, and exponentially, as \( h \to 0 \). To see this observe

\[
e(h) = \frac{1 - \gamma^{1/h}}{1 - \gamma}h^{p+1}
\]

\[
= \frac{1}{1 - \gamma}h^{p+1} - \frac{\gamma^{1/h}}{1 - \gamma}h^{p+1}.
\]

(17)

\[\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}\]
The first term clearly goes to zero. For the second term we can ignore the constant factor \(1/(1 - \gamma)\). The remaining term is

\[\gamma^{1/h} h^{p+1} = \frac{\gamma^n}{n^{p+1}}. \tag{18}\]

We are interested in the behavior of this expression as \(n\) tends to infinity. Applying the rule of L'Hôpital \(p + 1\) times we eventually get \((p + 1)!\) in the denominator, and \((\log \gamma)^{p+1}\gamma^n\) in the numerator. Thus the global error grows exponentially.

**Conclusions**

In this simple example, we encountered the following major issues in solving IVPs of ODEs:

1. The concept of a *fixed station limit* where the number of steps goes to infinity and the size of the individual steps goes to zero while the distance covered remains constant.

2. The concept of *error propagation* where the error generated by the \(n\)-th step depends on the entire preceding history, not just the \(n\)-th step-size.

3. We observed that with an amplification factor equal to 1 the order of the global error is one less than that of the local error. (We will see later that there must be one component of the error with an amplification factor equal to 1 if we wish to obtain any meaningful results at all. Thus the order of the global error will always be one less than that of the local error.)

4. We also observed that if the absolute value of the amplification factor is greater than 1 then the global error will grow exponentially as we decrease the step size. This phenomenon is called *instability*.

5. If the amplification factor is negative we obtain oscillating errors. Otherwise we obtain exponentially growing or
decaying errors. (Actually, we will also encounter complex amplification factors in which case the error will exhibit a low frequency oscillation with a superimposed exponential growth or decay.)

**Exercise**

Consider the error propagation in (4) for the more realistic case

\[ h = \frac{1}{n}, \quad \varepsilon = h^{p+1}, \quad \text{and} \quad \gamma = 1+h \quad \text{where} \quad h > 0 \quad (19) \]

and examine the behavior of \( e(h) \) as \( h \to 0 \).