Eigenvalue Problems

- all matrices $n \times n$ square, $x \in \mathbb{R}^n$, $\lambda$ is scalar (real, or possibly complex).
- $x \neq 0$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$ if
  \[ Ax = \lambda x \]

- An eigenvector is determined only up to a non-zero factor:
  \[ A(kx) = kAx = k(\lambda x) = \lambda(kx). \]

- Note that
  \[ Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0, \quad x \neq 0 \iff \det(A - \lambda I) = 0 \]

- The equation
  \[ \det(A - \lambda I) = 0 \]
  is the characteristic equation of $A$.

- The expression $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree $n$ with leading term $(-\lambda^n)$.

- why?

- $p(\lambda)$ is the characteristic polynomial of $A$.

- The eigenvalues are the roots of the characteristic polynomial. Thus $A$ has $n$ eigenvalues, properly counting multiplicity.

- Finding the eigenvalues by computing the characteristic polynomial and then finding the roots of that polynomial
is a very ill-conditioned process, since the roots of a polynomial are very sensitive with respect to small changes in the coefficients.

- However, the opposite way works very well. Start with a polynomial, normalize it to have leading coefficient $(-1)^n$, giving a polynomial $p$, then construct a matrix, called the **companion matrix** of $p$, which has $p$ as its characteristic polynomial, and then compute the eigenvalues of $A$.

- Suppose that the polynomial $p$ is given by

$$p(\lambda) = (-1)^n \left( \lambda^n - \alpha_{n-1}\lambda^{n-1} - \alpha_{n-2}\lambda^{n-2} - \ldots - \alpha_0 \right)$$

- The companion matrix $C$ of $p$ is defined by

$$C = \begin{bmatrix}
\alpha_{n-1} & \alpha_{n-2} & \ldots & \alpha_1 & \alpha_0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{bmatrix}$$

- Exercise: Show that

$$p(\lambda) = \det(C - \lambda I).$$

- The eigenvalues can be computed using standard software, most likely implementing the so-called **QR-Algorithm** (which is hugely complicated and mostly beyond our scope this semester).

- The matlab `roots` command find the roots of a polynomial that way.


**Relationships**

- Suppose we have \( n \) linearly independent eigenvectors \( x_i, i = 1, \ldots, n \) satisfying
  \[
  Ax_i = \lambda_i x_i, \quad i = 1, \ldots, n. \tag{1}
  \]

  Collect the eigenvalues and eigenvectors into matrices:

  \[
  X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}
  \]

- We can write (1) in matrix form as
  \[
  AX = X\Lambda
  \]
  which gives
  \[
  A = X\Lambda X^{-1} \quad \text{and} \quad \Lambda = X^{-1}AX.
  \]

- You can use these formulas to construct a matrix with given eigenvalues and eigenvectors which is sometimes useful.

- The rows of \( X^{-1} \) are the **left eigenvectors** of \( A \):
  \[
  X^{-1}A = \Lambda X^{-1}
  \]

- \( \Lambda = X^{-1}AX \) is an example of a **similarity transform** of \( A \).

- Two matrices \( A \) and \( B \) are **similar** if there exists a non-singular matrix \( T \) such that
  \[
  B = T^{-1}AT.
  \]
• Similar matrices have the same eigenvalues:

\[ Ax = \lambda x \implies B(T^{-1}x) = T^{-1}ATT^{-1}x \]
\[ = T^{-1}Ax \]
\[ = \lambda T^{-1}x \]

• The eigenvectors of \( B \) are \( T^{-1} \) times those of \( A \).

• If we have \( n \) linearly independent eigenvectors with real eigenvalues then we have a similarity transform to diagonal form.

• Of course the eigenvalues may be complex.

• However, the eigenvalues of a symmetric real matrix are real.

• Major issue: The eigenvectors of a matrix may not span the whole space of \( \mathbb{R}^n \). If they don’t the underlying matrix is said to be defective.

• Examples of defective matrices:
Suppose we do have a set of $n$ linearly independent eigenvectors. Let $X$ be the matrix that has those eigenvectors as its columns. Then

$$X^{-1}X = I, \quad X = [x_1, \ldots, x_n], \quad Y = X^{-1} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix}$$

and the left and right eigenvectors are biorthogonal:

$$y_i^T x_j = 0 \quad \text{if} \quad i \neq j$$

This can be useful for finding components corresponding to individual eigenvectors:
• Of course, it would be particularly nice to have an orthogonal set of eigenvectors. In that case we get

\[
AQ = QA \quad Q^T = Q^{-1}
\]

\[
\Lambda = Q^T AQ
\]

\[
A = Q\Lambda Q^T \quad \Rightarrow \quad A \text{ is symmetric!}
\]

• Only symmetric matrices have orthogonal sets of eigenvectors!

• The converse also holds: If \( A \) is symmetric it has an orthogonal set of eigenvectors. The proof is a bit complicated!
- Left and right eigenvectors corresponding to distinct eigenvalues are orthogonal:
• Generically eigenvalues are distinct. (If they are not we can apply an arbitrarily small perturbation to the matrix that will make the eigenvalues distinct.)

• But we may be in trouble if we have repeated eigenvalues.

• As mentioned above, a matrix is **defective** if its eigenvectors do not span all of \( \mathbb{R}^n \). Recall that a matrix is **singular** if zero is one of its eigenvalues.

• Singularity is unrelated to Defectiveness! Consider this table:

<table>
<thead>
<tr>
<th>singular</th>
<th>non-singular</th>
</tr>
</thead>
<tbody>
<tr>
<td>defective</td>
<td>[ ]</td>
</tr>
<tr>
<td></td>
<td>[ ]</td>
</tr>
<tr>
<td>non-defective</td>
<td>[ ]</td>
</tr>
</tbody>
</table>

• Is there a degree of defectiveness? Such as there is a degree of singularity, i.e., ill-conditioning?

• Yes! Discuss on Friday.
The Jordan Canonical Form

- Discuss if time permits.
- Two matrices $A$ and $B$ are similar if and only if they have the same Jordan (Canonical) Form (up to reordering the diagonal blocks).

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{bmatrix} \quad \text{where} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

- The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial. The geometric multiplicity of an eigenvalue is the number of linearly independent corresponding eigenvectors.
- Each Jordan block $J_i$ corresponds to one eigenvector.
- Thus the dimension of the space spanned by the eigenvectors of $J$ equals $k$.
- The Jordan Canonical Form is not computable in approximate arithmetic.