Norms and Error Analysis

Vector and Matrix Norms

• We are familiar with the norm of a vector $x \in \mathbb{R}^n$:

$$\|x\| = \|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$  

• This is also called the 2-norm, the standard norm, the Euclidean Norm, the magnitude, or the Euclidean length, of $x$.

• But it is useful to generalize the concept. Henceforth we will think of $\|x\|_2$ as only one of infinitely many norms.

• This is a standard generalization technique in mathematics:
  – Decide and list what properties are important.
  – Find other objects with those properties.

• So what are the key properties of a norm?

$$\|x\| \geq 0, \quad \|x\| = 0 \Rightarrow x = 0,$$

$$\|\alpha x\| = |\alpha| \|x\|, \quad \|x + y\| \leq \|x\| + \|y\|.$$
• **Definition:** A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) that associates a number \( \|x\| \) with a vector \( x \) is called a (vector) **norm** if the following four properties hold for all vectors \( x, y \) and scalars \( k \):

1. \( \|x\| \geq 0 \)
2. \( \|x\| = 0 \implies x = 0 \)
3. \( \|kx\| = |k|\|x\| \)
4. \( \|x + y\| \leq \|x\| + \|y\| \)

• Property 4 is called the **triangle inequality**.

• If property 2 is missing then the function \( \|x\| \) is called a **semi-norm**.
Examples of Vector Norms

• Throughout let

\[ x = [x_i]_{i=1,\ldots,n} \in \mathbb{R}^n. \]

• The \( p \)-norm, for \( p \geq 1 \), is defined by

\[ \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}. \]

Special cases of the \( p \)-norm are

– the \textbf{1-norm}, \( p = 1 \):

\[ \|x\|_1 = \sum_{i=1}^{n} |x_i| \quad \|\begin{bmatrix} 3 \\ -4 \end{bmatrix}\|_1 = 7 \]

– the standard norm, etc., \( p = 2 \):

\[ \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \quad \|\begin{bmatrix} 3 \\ -4 \end{bmatrix}\|_2 = 5 \]

– the \textbf{infinity} or \textbf{Chebychev norm}, \( p = \infty \):

\[ \|x\|_\infty = \max_{i=1,\ldots,n} |x_i| \quad \|\begin{bmatrix} 3 \\ -4 \end{bmatrix}\|_\infty = 4 \quad (1) \]

• \textbf{Query}: Why is there no sum in (1)?

• The \( p \)-norms satisfy the \textbf{Hölder Inequality}. Suppose

\[ \frac{1}{p} + \frac{1}{q} = 1 \]

Then

\[ |x^T y| \leq \|x\|_p \|y\|_q \]
• In the special case that $p = q = 2$ this turns into the **Cauchy-Schwarz Inequality**:

$$|x^T y| \leq \|x\|_2 \|y\|_2$$  \hspace{1cm} (2)

• Of course we know the stronger result that

$$x^T y = \|x\|_2 \|y\|_2 \cos \theta$$  \hspace{1cm} (3)

where $\theta$ is the angle formed by $x$ and $y$. Clearly (3) implies (2).

• Another useful norm is a **weighted $p$-norm**. Suppose we have a vector $w$ of weights:

$$w \in \mathbb{R}^n, \quad w_i > 0, \quad i = 1, \ldots, n.$$  

$$\|x\|_{w,p} = \left( \sum_{i=1}^{n} w_i |x_i|^p \right)^{1/p}$$

• A weighted $p$-norm is a special case of the norm $\| \cdot \|_A$ defined by a given norm $\| \cdot \|$ and a non-singular matrix $A$:

$$\|x\|_A = \|Ax\|$$

• **Query**: Why does $A$ have to be non-singular?

• **Exercise**: Show that all of the above examples do in fact define a norm.

• **Exercise**: Prove the validity of the Hölder and Cauchy-Schwarz inequalities.

• Choosing among norms is largely a matter of convenience. On $\mathbb{R}^n$, and in fact on all finite-dimensional vector spaces, all norms are equivalent in the sense that if $\| \cdot \|$ and $\| \cdot \|_*$ are two given norms then there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \|x\| \leq \|x\|_* \leq c_2 \|x\|$$

for all $x \in \mathbb{R}^n$.

• **Exercise**: Verify this statement. (This is a bit subtle.)
Errors

• Norms can be used to measure errors. Suppose $x$ and $\hat{x}$ are vectors in $\mathbb{R}^n$ where we think of $\hat{x}$ as an approximation of $x$:

$$\hat{x} \approx x$$

• Then $\|x - \hat{x}\|$ is the absolute error in $\hat{x}$ (with respect to the norm $\| \cdot \|$)

• Similarly, $\frac{\|x - \hat{x}\|}{\|x\|}$ is the relative error.

Convergence

• Norms can be used to define and analyze the convergence of a sequence of vectors. We say that the sequence

$$x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, \ldots$$

converges to $x$ if

$$\lim_{k \to \infty} \|x^{(k)} - x\| = 0.$$

• Because of the equivalence of norms the convergence of the sequence is independent of the choice of the norm used for the analysis.
Matrix Norms

- The space of $m \times n$ matrices is isomorphic to $\mathbb{R}^{mn}$ and we can apply the same definition as we did for vectors:

**Definition:** A function $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ that associates a number $\|A\|$ with a matrix $A$ is called a **matrix norm** if the following four properties hold for all $m \times n$ matrices $A$, $B$ and scalars $k$:

1. $\|A\| \geq 0$
2. $\|A\| = 0 \implies A = 0$
3. $\|kA\| = |k|\|A\|$  \hspace{1cm} (4)
4. $\|A + B\| \leq \|A\| + \|B\|$

- For example, the **Frobenius Norm** $\|A\|_F$ of a matrix $A$ is defined by thinking of the matrix as a vector and applying the vector 2-norm:

$$\|A\|_F = \sqrt{\sum_i \sum_j a_{ij}^2}.$$

The problem with thinking of the matrix as a giant vector is that it does not tell us what happens to the norm when we multiply matrices.

- The link between norms and matrix products is provided by the concept of an **induced matrix norm** or **operator norm**:

  - given vector norms in $\mathbb{R}^m$ and $\mathbb{R}^n$ we define the induced matrix norm of an $m \times n$ matrix $A$ as

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$
• Note that the vector norms operate on different spaces if the matrix $A$ is rectangular.

• **Exercise**: Show that every induced matrix norm satisfies the conditions listed in (4).

• The great advantage of using an induced matrix norm is that we get the additional property that the norm of the product is never larger than the product of the norms. More formally:

• **Product Property**: Let $\| \cdot \|$ denote both the pertinent vector norms and the induced matrix norms. Then, for all matrices $A$ and $B$ and vectors $x$:

$$
\|Ax\| \leq \|A\|\|x\| \quad \text{and} \quad \|AB\| \leq \|A\|\|B\| \quad \text{(5)}
$$

• **Query**: Why do we use the plural “norms” in the above statement?

• The properties in (5) are easy to verify. The first follows straight from the definition:

$$
\|Ax\| = \frac{\|Ax\|}{\|x\|}\|x\| \leq \|A\|\|x\|.
$$

The second follows from the first. We have, for some vector $x$ with

$$
\|x\| = 1,
$$

that

$$
\|AB\| = \|ABx\|
$$

$$
\leq \|A\|\|Bx\|
$$

$$
\leq \|A\|\|B\|\|x\|
$$

$$
= \|A\|\|B\|
$$

**The Infinity Norm**

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Some induced matrix norms are much easier to compute than others. For example, the infinity- or Chebychev-norm of an $m \times n$ matrix $A$ is simply the maximum row sum:

$$
\| A \|_\infty = \max_{\|x\|_\infty = 1} \| Ax \|_\infty = \max_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|.
$$

(6)

Example:

$$
\left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \right\|_\infty = 8
$$

To see (6) let $S$ denote the maximum row sum:

$$
S = \max_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}| = \sum_{j=1}^{n} |a_{kj}|
$$

for some $k$. Thus $k$ is the index of some row that has the maximum sum. If there are several such rows we pick any particular one of them.

We need to show that $\| Ax \|_\infty \leq S$ for all vectors $x$ that satisfy $\|x\|_\infty = 1$ and that there exist some particular vector $x$ with $\|x\|_\infty = 1$ such that $\| Ax \|_\infty \geq S$.

To see the first inequality let $x$ be such that

$$
\| x \|_\infty \leq 1,
$$

i.e.,

$$
|x_i| \leq 1, \quad i = 1, \ldots, n.
$$
Then
\[ \|Ax\|_\infty = \max_{i=1,\ldots,m} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \]
\[ \leq \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}x_j| \]
\[ = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}| |x_j| \]
\[ \leq \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}| \]
\[ = S \]

- On the other hand, with
  \[ s = [\text{sign}a_{kj}] \]
  where
  \[ \text{sign}z = \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \\ 0 & \text{if } z = 0 \end{cases} \]
  we have
  \[ \|As\|_\infty = \max_{i=1\ldots,n} \left| \sum_{j=1}^{n} a_{ij}\text{sign}a_{kj} \right| \]
  \[ \geq \left| \sum_{j=1}^{n} a_{kj}\text{sign}a_{kj} \right| \]
  \[ = \sum_{j=1}^{n} |a_{kj}| \]
  \[ = S \]

- For example, for the previously considered matrix
  \[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ |x| = x \cdot \text{sign} x \]
the maximum row sum is 8 and occurs in row 2. The vector of signs is $s = [1, -1, -1]^T$ and indeed $\|As\|_\infty = 8$:

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ -4 \\ 8 \\ -1 \end{bmatrix}$$

- **Exercise:** Use the same idea to show that the 1-norm of a matrix is the maximum column sum:

$$\|A\|_1 = \max_{j=1}^n \sum_{i=1}^m |a_{ij}|.$$  

- Compare this with the 2-norm of a matrix. It is given by

$$\|A\|_2 = \max \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\rho(A^T A)}$$

where $\rho(B)$ is the **spectral radius** of a square matrix $B$. The spectral radius is the maximum of the absolute values of the eigenvalues.

- This is much harder to compute than the 1 or infinity norm. A straightforward calculation would require $O(n^2)$ operations for the 1 or infinity norm and $O(n^3)$ operations for the computation of the 2-norm of $A$.

- **Exercise:** Show that for any square matrix $A$ and any induced matrix norm $\| \cdot \|$  

$$\|A\| \geq \rho(A)$$

- **Exercise:** Show that the Frobenius norm is not induced by any vector norm.

- Our next task is to use the ideas we discussed today to analyze the role of errors in solving a linear system.
Sensitivity Analysis for $Ax = b$

- Consider the square linear system

$$Ax = b$$ (7)

- How do errors
  - in $A$
  - in $b$
  - in the solution process affect the accuracy of the computed solution?

- There are two main types of error analysis. In **Forward Error Analysis** we examine how errors in the data ($A$ or $b$) propagate through the specific method used for solving $Ax = b$ and effect the solution. A classic example of this technique is **Interval Analysis** (available, for example, in the Matlab INTLAB package) where numbers are represented as intervals (containing the number being represented).

- The second type, which is less specific, but more generally applicable, is **Backward Error Analysis** where the numerical solution is thought of not as the approximate solution of an exactly given problem, but as the exact solution of an approximately given problem. The results so obtained are independent of the numerical method used to solve the mathematical problem.

- In these notes we explore this idea in the context of the linear system (7).

- Suppose that, by some means or other, we obtain an approximate solution $\hat{x}$ of (7) that contains an error $e$, i.e.,

$$\hat{x} = x - e.$$ (8)

- We know neither $x$ nor $e$, but we can compute the residual

$$r = b - A\hat{x}.$$ (9)
• It is crucial that \( e \) and \( r \) are related in the same way as \( x \) and \( b \):

\[
Ae = A(x - \hat{x}) = Ax - A\hat{x} = b - A\hat{x} = r. \quad (10)
\]

This illustrates a general principle: for any linear problem, the error satisfies the same equation as the solution, except that the right hand side is replaced by the residual. This applies not just to linear equations but also to linear operator equations like linear differential and integral equations.

• This principal is so important that I’ll derive it in a different, perhaps more intuitive, way. Start with

\[
A\hat{x} = A(x - e) = Ax - r = b - r. \quad (11)
\]

Then

we have \( Ax = b \)
and \( A\hat{x} = b - r \)
Take the difference:

\[
Ae = r. \quad (12)
\]

• The essence of backward error analysis is that we consider \( \hat{x} \) the exact solution of the perturbed problem \( A\hat{x} = b - r \) rather than the approximate solution of the problem \( Ax = b \). So we are addressing the question of how perturbations of the right hand side effect the solution of the original problem (7).

• So how do we get a handle on this? We are interested in the error, or its norm. We can afford a large error if the solution itself is large. So we are asking about the relative error \( \|e\|/\|x\| \) (which we don’t know) and how it relates to the relative residual \( \|r\|/\|b\| \) (which we can compute).

• One way to approach the matter is simply to write down what we know, and then go from there. A creative step is to invoke the inverse of \( A \), even though of course we never actually invert a matrix.
• Using any vector norm and the induced matrix norm, we obtain:

\[
\begin{align*}
Ax &= b & \|b\| \leq \|A\|\|x\| \quad [1] \\
A^{-1}b &= x & \|x\| \leq \|A^{-1}\|\|b\| \quad [2] \\
Ae &= r & \|r\| \leq \|A\|\|e\| \quad [3] \\
A^{-1}r &= e & \|e\| \leq \|A^{-1}\|\|r\| \quad [4]
\end{align*}
\]

(13)

• We can combine two inequalities in (13) by dividing the smaller of one by the larger of the other, and the larger of the one by the smaller of the other. In particular, if we divide the larger side of [4] by the smaller of [1], and the smaller of [4] by the larger of [1], we obtain

\[
\frac{\|e\|}{\|A\|\|x\|} \leq \frac{\|A^{-1}\|\|r\|}{\|b\|}
\]

which can be rewritten as

\[
\frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|r\|}{\|b\|}.
\]

(15)

• Similarly, we obtain by combining [2] and [3]:

\[
\frac{1}{\|A\|\|A^{-1}\|} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|}.
\]

(16)

• Combining (15) and (16) we obtain

\[
\frac{1}{\|A\|\|A^{-1}\|} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|r\|}{\|b\|}
\]

(17)

These inequalities merit deep study!

Notes:

1. The expression \(\|A\|\|A^{-1}\|\) is called the condition number of \(A\) with the respect to the underlying vector norm.
Unless it’s specified otherwise the norm is usually the 2-norm.

2. The matrix \( A \), and the linear system (7), are said to be **ill conditioned** if \( \| A \| \| A^{-1} \| \) is large, and **well conditioned** if \( \| A \| \| A^{-1} \| \) is small. The meaning of “large” and “small” depends on the context and will become clearer in the following notes.

3. The relative residual will usually be greater than the round-off unit, i.e., the smallest positive number \( \tau \) for which the machine recognizes that \( 1 + \tau \) is greater than 1. On a Unix system, \( \tau \approx 10^{-16} \). Thus a solution cannot be expected to have more than 16 correct digits. If the condition number is \( 10^p \) then this means that the solution may have no more than \( 16 - p \) correct digits. If the condition number is \( 10^{16} \) then the relative error may be as large as 1, the error will be as large as the solution, and therefore we won’t have a solution.

4. We never compute a matrix inverse, and so it’s not trivial to compute or approximate \( \| A \| \| A^{-1} \| \). An effective approximation, implemented in LAPACK and MATLAB is described in the beautiful paper


5. While the condition number does depend on the norm being used, it can be large regardless of the norm. Let \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) the largest and smallest of the absolute values of the eigenvalues of \( A \). Since for any induced matrix norm, \( \| A \| \geq \lambda_{\text{max}} \) and \( \| A^{-1} \| \geq 1/\lambda_{\text{min}} \), we have

\[
\| A \| \| A^{-1} \| \geq \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \tag{18}
\]

6. It’s clear (e.g., from (18)), that \( \| A \| \| A^{-1} \| \geq 1 \). A matrix with condition number 1 is as well conditioned as it can be.

7. The condition number (with respect to the 2-norm) of an orthogonal matrix is 1. In other words, orthogonal matrices are as well conditioned as possible.
8. A matrix is singular if and only if zero is one of its eigenvalues. Consider a non-singular matrix $A$ where one eigenvalue remains constant, and another approaches zero. In that sense, $A$ approaches singularity. As it does so, according to (18), the condition number approaches infinity.

9. This illustrates a general point: If there is a mathematical singularity you expect numerical difficulties if you are close to that singularity.

10. The inequalities (18) are sharp. It is possible, for any vector norm and its corresponding induced matrix norm, and for either of the inequalities in (18), to find vectors $x$ and $e$ such that the inequality is satisfied with equality. It’s a good exercise to verify this statement.

11. The inequalities (17) say that the relative error may be as large as the relative residual multiplied with the condition number, or as small as the relative residual divided by the condition number.

12. However, one can do a probabilistic analysis that shows that with a high probability the right hand inequality in (17) is satisfied with close to equality. In other words, the quotient

$$\frac{\|e\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|r\|}{\|b\|}$$

(19)

will be less than, but close to 1. For details see the above mentioned paper by Cline et al.

13. The inequalities (17) are independent of the method that is used to solve (7). This means that once you have an ill-conditioned linear system, there is little you can do. The key to handling ill-conditioning is to avoid it, not to fight it. We will see various ways of doing this in various contexts as we proceed through the semester. The example below illustrates this idea.

14. The condition number of a $1 \times 1$ “matrix” is of course 1 (why?). The formal view of a diagonal matrix is that it may be ill-conditioned. For example, the 2-norm condition
number of the matrix
\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 10^p \end{bmatrix} \] (20)
is $10^p$ and can be arbitrarily large. However, linear equations with a diagonal matrix do not interact at all. It is more fruitful to think of a diagonal “system” $Dx = b$ as a set of $n$ separate single equations, all of which have condition number 1.

**Exercise.** In the above analysis we interpret the error $e$ as being caused by a perturbation $r$ of the right hand side $e$. How about a perturbation of the coefficient matrix? Consider the system
\[(A - E)(x - e) = b\] (21)
and show that
\[ \frac{\|e\|}{\|x - e\|} \leq \|A\|\|A^{-1}\|\|E\|\|A\|. \] (22)

Thus we have the same sort of result: the relative error in the solution may be as large as the condition number multiplied with the relative error in the coefficient matrix. Again, the inequality is sharp.

**An Example**

Ill-conditioning can easily occur when doing things that come naturally.

Suppose we want to approximate a function $f$ with a polynomial $p$ such that
\[ \int_0^1 (f(x) - p(x))^2 dx = \min. \] (23)

Writing
\[ p(x) = \sum_{j=0}^n \alpha_j x^j \] (24)
we obtain the minimization problem

\[ F(\alpha_0, \alpha_1, \ldots, \alpha_n) = \int_0^1 \left( f(x) - \sum_{j=0}^n \alpha_j x^j \right)^2 \, dx = \min \]

(25)

To solve this problem we proceed as usual, by differentiating with respect to variables, and setting the gradient equal to zero. This gives

\[ \frac{\partial}{\partial \alpha_i} F = -\int_0^1 \left( f(x) - \sum_{j=0}^n \alpha_j x^j \right) x^i \, dx = 0, \quad i = 0, \ldots, n \]

(26)

This can be written as the \((n+1) \times (n+1)\) linear system

\[ H_{n+1} x = b \]

(27)

where \(H_{n+1}\) is the \((n+1) \times (n+1)\) **Hilbert Matrix** with entries

\[ h_{ij} = \int_0^1 x^i x^j \, dx = \frac{1}{i + j + 1}, \quad i, j = 0, \ldots, n \]

(28)

Thus, for example,

\[ H_4 = \begin{bmatrix}
    1/1 & 1/2 & 1/3 & 1/4 & 1/5 \\
    1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\
    1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\
    1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\
    1/5 & 1/6 & 1/7 & 1/8 & 1/9
\end{bmatrix} \]

(29)

The Hilbert matrix is obviously symmetric, and it’s a good exercise to show that it is positive definite. It also happens to have an inverse where all entries are integer. However, floating point computations with the Hilbert matrix are
essentially impossible, due to its poor conditioning, as illustrated in this Table:

\[
\begin{array}{cccccc}
\|H_n\|_2 \|H_n^{-1}\|_2 : & 2 & 3 & 6 & 10 & 15 \\
19 & 524 & 1.5 \times 10^6 & 1.6 \times 10^{13} & 2.5 \times 10^{17} & (30)
\end{array}
\]

Thus we’ll loose about 17 of 16 digits when computing with the 15 × 15 Hilbert matrix!

It is not surprising that the Hilbert matrix is ill-conditioned, because its rows all look the same! If they were identical

\textbf{Figure 1.} Monomials $x^n$, $n = 0, 1, \ldots, 10$. 

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the matrix would be singular, if they all look similar the matrix is ill-conditioned (despite being positive definite).

It’s also no surprise that our basic approach gives rise to an ill-conditioned system, because the monomials in (24) all have the same shape. To the seasoned numerical analysts it makes no sense to express a polynomial, that can have many oscillations, in terms of a bunch of basis functions that all have the same shape, and no oscillations at all, as in Figure 1.

So what can we do? The idea is to use polynomial basis functions that do not have these deficiencies. In this case...
one should use **shifted Legendre Polynomials**. Letting

$$<f, g> = \int_0^1 f(x)g(x)\,dx$$  \hfill (31)

these are defined$^{-1} -$ as follows

- $P_0(x) = 1$
- $P_1(x) = x - a_1$
- $P_n(x) = (x - a_n)P_{n-1} - b_nP_{n-2}$

where $a_n = \frac{<xP_{n-1}, P_{n-1}>}{<P_{n-1}, P_{n-1}>}$ and $b_n = \frac{<xP_{n-1}, P_{n-2}>}{<P_{n-2}, P_{n-2}>}$

(32)

This gives (exercise)

- $P_0(x) = 1$
- $P_1(x) = 2x - 1$
- $P_2(x) = 6x^2 - 6x + 1$
- $P_3(x) = 20x^3 - 30x^2 + 12x - 1$
- $P_4(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1$
- $P_5(x) = 252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1$

$$\ldots$$

(33)

With these polynomials one does not need to solve a linear system at all, since the resulting matrix, corresponding to the Hilbert matrix, is **diagonal** (exercise). In fact, the solution of the problem (23) is, simply,

$$p(x) = \sum_{j=0}^{n} \frac{<f, P_j>}{<P_j, P_j>} P_j(x).$$  \hfill (34)

---

- see, e.g., Cheney, Introduction to Approximation Theory, p. 107
The basis functions $P_i$ all look very different, as illustrated in Figure 2.

So here is the moral of our story:

**Don’t fight ill-conditioning, avoid it!**

(35)