Radial Basis Functions

- **References** There is a vast number of books on radial basis functions. Two good places to start are:

Scattered Data

- We consider the following interpolation problem:
- Given data
  \[(x_i, y_i, z_i), \quad i = 1, \ldots, n\]
  find an interpolant \(s\) such that
  \[s(x_i, y_i) = z_i, \quad i = 1, \ldots, n\]
  \[s(x, y) = \sum_{i=1}^{n} \lambda_i \phi_i(x, y)\]
- There are two basic approaches: **triangle based** and **meshfree**.
- Let’s first consider meshfree methods.
- We saw that if we want to interpolate with polynomials (in two variables) there may not be a solution.
- Indeed, no matter how we pick the basis functions independently of the data, there are data where the Vandermonde matrix is singular.
- So we must pick the basis functions dependent on the data sites \((x_i, y_i)\).
• The most successful choice of basis functions gives rise to radial basis functions, or, simply, (RBF)

• Radial basis functions have the additional advantage that they work in any dimension. However, we'll describe them in terms of two variables.

• A radial basis function is a function of the form

\[ \phi_i(x, y) = g(\| (x, y) - (x_i, y_i) \|) \]

where \( g \) is a given univariate function and \( \| \cdot \| \) is the ordinary 2-norm.

Examples

\[ \phi(x) = g(\| x - x_i \|) \]

\[ x \in \mathbb{R}^n \]

\[ \| x \| = \sqrt{x^T x} = \sum x_i^2 \]

\[ \| x \| = \max |x_i| \]

\[ \| x \| = \sqrt{x^T A x} \]

\[ A = A^T \text{ pos def.} \]

\[ \text{you can use other norms} \]

**Figure 1.** Hardy Multiquadrics, \( h = 1 \).

• Hardy Multiquadrics are defined by

\[ g(t) = \sqrt{h^2 + t^2} \]
or
\[ g(t) = \frac{1}{\sqrt{h^2 + t^2}}. \]

The parameter \( h \) is a shape parameter that can be chosen based on the problem.

- Rolland L. Hardy was a Civil Engineer at Iowa State University who pioneered the subject and introduced RBF in 1970. His paper *Multiquadric Equations of Topography and Other Irregular Surfaces*, Journal of Geophysical Research, v. 76, No. 9, March 1971, pp. 1905–1915, is available online.

![Figure 2: Duchon Thin Plate Splines](image)

**Figure 2.** Duchon Thin Plate Splines.

- **Duchon Thin Plate Splines**

  \[ g(t) = t^2 \log t. \]

  These provide a natural analog to univariate Cubic Splines. The interpolant based on them minimizes the clamped elastic plate functional

  \[
  \int \int_{\mathbb{R}^2} s_{xx}^2(x, y) + 2s_{xy}^2(x, y) + s_{yy}^2(x, y) \, dx \, dy = \min
  \]

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  1 var.

  \[ \int_a^b (s''(x))^2 \, dx = \min \]
subject to the interpolation requirement.


- **Gaussian** RBF use

  \[ g(t) = e^{-(\epsilon t)^2} \]

  where \( \epsilon \) is a suitably chosen tuning parameter.

  ![Figure 3. Gaussian RBF \( \epsilon = 1 \).](image)

### Concepts and Ideas

- There is nothing special about having two variables. If \( x \in \mathbb{R}^k \) and we are given points \( x_i \in \mathbb{R}^k, i = 1, 2, \ldots, n \)
we can use
\[
s(x) = \sum_{i=1}^{n} \alpha_i g(\|x - x_i\|)
\]
to interpolate:
\[
s(x_i) = z_i.
\]

• We can find the coefficients by solving the Vandermonde System (1).
• The Vandermonde System can be shown to be non-singular for many choices of \( g \), including those listed in these notes.
• A major issue is that the Vandermonde system is often ill-conditioned.
• The Vandermonde system for the choices shown so far is dense.
• The scheme is global.
• However, this problem can be fixed by using locally supported RBF, such as
\[
g(x) = \begin{cases} 
\exp \left( \frac{-1}{1-(\frac{x}{r})^2} \right) & \text{if } -r < x < r \\
0 & \text{else}
\end{cases}
\]
• Again, \( r \) is a shape parameter.
• The RBF so far discussed do not reproduce polynomials.
• However one can force polynomial precision of arbitrary degree by a cool trick.
• A polynomial of degree \( d \) in \( k \) variables \( x_1, x_2, \ldots, x_k \), is of the form
\[
p_d(x_1, \ldots, x_k) = \sum_{i_1+i_2+\ldots+i_k \leq d} \alpha_{i_1,\ldots,i_k} x_1^{i_1} \ldots x_k^{i_k}
\]
• Query: How many parameters does \( p \) have?
Figure 4. Local RBF $r = 1$.

- To force polynomial precision of degree $d$ write

$$s(x) = \sum_{i=1}^{n} \alpha_i g(\|x - x_i\|) + p_d(x)$$

where $x \in \mathbb{R}^k$ is the vector $[x_1, \ldots, x_k]$ of variables.

- To determine the parameters in $s$ impose the interpolation conditions and the additional conditions

$$\sum_{i=1}^{n} \alpha_i q(x_i) = 0$$

for all polynomials $q$ of degree $\leq d$.

Clearly (why) if $z_i = p(x)$ for some polynomial $p$ of degree $d$ then

$$s(x) = p(x).$$
• Similarly, we could force reproduction of other types of functions as well.

Summary

RBF work in a general number of variables.

• They typically have a shape parameter whose choice can be crucial.

• The Vandermonde matrix is usually dense, and often ill-conditioned.

• The basis functions depend on the data.

• There are many ways to tweak the interpolant according to the circumstances of the underlying problem.

• Finding cardinal basis functions is not as straightforward as for other schemes we have considered.