Bivariate Interpolation

- Interpolation still means exact reproduction of data.
- The word bivariate refers to having two independent variables.
- Also common are the words trivariate (three independent variables), multivariate (more than one independent variable), and univariate (one independent variable).
- Much of what we’ll do in the bivariate context carries over to the more general multivariate context.
- However, there is a vast difference between univariate and multivariate contexts.
- We’ll consider five different kinds of interpolation:
  - Tensor Product
  - Polynomial
  - Transfinite
  - piecewise polynomial on triangles
  - meshless, or radial basis functions.

Tensor Product

- The phrase tensor product refers to a situation where the data lie on a rectangular grid, and we basically do univariate interpolation twice.
- Suppose we are given data

\[(x_i, y_j, f(x_i, y_j)), \quad i = 0, \ldots, m, \quad j = 0, \ldots, n\]
• We want an interpolant \( p \) satisfying

\[
p(x_i, y_j) = f(x_i, y_j)
\]

for all \( i \) and \( j \).

• An important basic idea is to consider operators that do the job in one of the variables and then compose, or otherwise combine them.

• Consider polynomial interpolation. Define

\[
P_x f(x, y) = \sum_{i=0}^{m} f(x_i, y)L_i(x) \quad \text{where} \quad L_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.
\]

Clearly \( P_x f \) interpolates in \( x \) for all values of \( y \). Similarly, we define

\[
P_y f(x, y) = \sum_{j=0}^{n} f(x, y_j)M_j(x) \quad \text{where} \quad M_j(y) = \frac{\prod_{i \neq j}(y - y_i)}{\prod_{i \neq j}(y_j - y_i)}.
\]
• The overall interpolant is then simply the composition of those two interpolants.

\[ P f = P_x P_y f \]

\[ = P_x \sum_{j=0}^{n} f(x, y_j) M_j(y) \]

\[ = \sum_{i=0}^{m} \left( \sum_{j=0}^{n} f(x_i, y_j) M_j(y) \right) L_i(x) \]

\[ = \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i, y_j) M_j(y) L_i(x) \]

\[ = p(x, y) \]

• Clearly, \( p \) interpolates at all points \( (x_i, y_j) \).
Bilinear Interpolation

- For simplicity, consider the unit square.
- Let
  
  \[ P_x f(x, y) = (1 - x)f(0, y) + xf(1, y) \]
  and
  
  \[ P_y f(x, y) = (1 - y)f(x, 0) + yf(x, 1) \]  

- Clearly \( P_x \) interpolates at all points \((0, y)\) and \( P_y \) interpolates all points \((x, 0)\) and \((x, 1)\).
- The composition gives the bilinear interpolant to the function values at the vertices of the unit square.

  \[
P f(x, y) = P_x P_y f(x, y) = P_x ((1 - y)f(x, 0) + yf(x, 1)) = (1 - x)((1 - y)f(0, 0) + yf(0, 1)) + x((1 - y)f(1, 0) + yf(1, 1)) = (1 - x)(1 - y)f(0, 0) + (1 - x) y f(0, 1) + x(1 - y)f(1, 0) + xyf(1, 1) = A + Bx + Cy + Dxy
  \]

- The interpolant is bilinear which means that setting one variable to a constant gives a function that is linear in the other variable.
- Note that the restriction of the bilinear function to a line, such as \( y = x \), in general gives a quadratic function.
Transfinite Interpolation

- We illustrate the idea by interpolating everywhere on the boundary of the unit square.

- This is not as far fetched as it may sound. For example, the data could come from a grid that approximate the shape of a vehicle such as a car or aircraft. The curves along the grid lines could have been designed by some univariate scheme. Or we might have a vehicle, such as a survey ship, move along a line and collect data densely along the line.

- Let $P_x$ and $P_y$ be the linear operators defined in (1).

- The following bilinearly blended Coon’s patch interpolates on the boundary of the unit square. It is also called the Boolean Sum of the operators $P_x$ and $P_y$.

$$Qf = P_x f + P_y f - P_x P_y f$$

$$= (1 - x)f(0, y) + xf(1, y) + (1 - y)f(x, 0) + yf(x, 1) - (1 - x)(1 - y)f(0, 0) - (1 - x) yf(0, 1) - x(1 - y)f(1, 0) - xyf(1, 1)$$

- Check interpolation

- Exercise: try cubic Hermite interpolation on the unit square, finite and transfinite.
Scattered Data

- The phrase “scattered data” means that the data sites do not form a rectangular grid.
- So we are given data
  \[(x_i, y_i, z_i), \quad i = 1, 2, \ldots, n\]
  and we want an interpolant \(P\) such that
  \[P(x_i, y_i) = z_i, \quad i = 1, 2, \ldots, n.\]
- The natural first step would be to attempt interpolation by a polynomial.
- The immediate stumbling block is that there is no unique polynomial associated with the number of data.
- A bivariate polynomial \(p\) of degree \(d\) is defined by
  \[p(x, y) = \sum_{i+j \leq d} \alpha_{ij} x^i y^j\]
• How many parameters does a polynomial of degree $d$ have?
Linear Interpolation

• Writing the interpolant as

\[ L(x, y) = \alpha_{10}x + \alpha_{01}y + \alpha_{00} \]

we get the Vandermonde system

\[
\begin{bmatrix}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  x_3 & y_3 & 1
\end{bmatrix}
\begin{bmatrix}
  \alpha_{10} \\
  \alpha_{01} \\
  \alpha_{00}
\end{bmatrix}
= 
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3
\end{bmatrix}
\]

• Under what circumstances is the Vandermonde matrix singular?
Quadratic Interpolation

• Supposing our interpolant is

\[ Q(x, y) = \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{10}x + \alpha_{01}y + \alpha_{00} \]

we get the Vandermonde System

\[
\begin{bmatrix}
    x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\
    x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\
    x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\
    x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\
    x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \\
    x_6^2 & x_6y_6 & y_6^2 & x_6 & y_6 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    \alpha_{20} \\
    \alpha_{11} \\
    \alpha_{02} \\
    \alpha_{10} \\
    \alpha_{01} \\
    \alpha_{00} \\
\end{bmatrix}
=
\begin{bmatrix}
    z_1 \\
    z_2 \\
    z_3 \\
    z_4 \\
    z_5 \\
    z_6 \\
\end{bmatrix}
\]

• Under what circumstances is the Vandermonde matrix singular?
• Maybe we shouldn’t use polynomials. Maybe there are better functions.

• Let

\[ p(x, y) = \sum_{i=1}^{n} \alpha_i \phi_i(x, y). \]

• How do we pick the basis functions \( \phi_i \)?

It turns out that no matter how we pick them, there are Lagrange Interpolation problems with distinct data sites such that the Vandermonde matrix is singular.