Remaining Events

— The University has scheduled our final exam for

Tuesday, April 28, 2020

It’s format will be like that of the old exam I handed out in the past, except that the questions will be different. So you’ll be asked to answer \( n \) out of \( m \) questions where \( n \) is about half of \( m \). The exam will be designed such that two hours is about the right length of time to answer your chosen questions but you can use more time if you like. Email me a pdf with your answers some time on April 28. The answer need not be typeset, although of course you are welcome to typeset them if you like. You should work on the exam by yourself, without using notes, literature, or electronic devices.

— I should be able to send you your scores and your final grade within one or two days of the final exam.

— I will be available for consultation via zoom by appointment. Send me an email if you want to talk with me, and we will probably be able to schedule a meeting very quickly.
Summary

or Numerical Analysis in 100 Easy Steps

1. Rules to live by

1.1 Asking the right questions is more important than knowing some right answers.

1.2 If you don’t understand a question you won’t be able to answer it. Don’t even try. Figure out first what the question is asking.

1.3 Always have expectations. (If they are met, good. Otherwise find the error or figure out what it is that you did not understand.) Think about your expectations, and make them explicit, before starting to work on a problem.

1.4 Always check your answers.

1.5 Everybody makes mistakes. That’s OK. It’s not OK not to catch those mistakes.

1.6 To solve a difficult problem build a hierarchy of simpler problems leading up to it. Never hesitate to make simplifying assumptions. Worry about those assumptions only after you answer the simpler question.

1.7 First, assume all horses are spherical.

1.8 Even though a problem may be difficult and take a lot of time for its solution, once you solve it the next similar problem will be easier.

1.9 Match the number of parameters and the number of conditions.

\[\text{-1-} \] This phrase was invented by John Lund, a student in an earlier incarnation of this class. He volunteered to draw up a list (which overlapped with this list, but was different) and present it to that class. His main occupation, however, was serving as an officer in the US Navy. He once told me that his favorite activity was “driving a nuclear submarine”.

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1.10 Choose the right level of abstraction and generality. (Example: the interpolant of symmetric data is symmetric.)

1.11 Go back to the definition.

1.12 Focus on understanding rather than on recipes.

1.13 Understanding something means being able to explain it in simpler terms and relating it in various ways to other things.

1.14 Make sure there is a great deal of redundancy in your understanding.

1.15 To succeed in mathematics you must master its language.

1.16 If something has a lot of names it must be important.
Quotes

2. My first thought when I find myself in a room full of manure is “there must be a pony here somewhere”. (Ronald Reagan)

3. The purpose of computing is insight, not numbers (Hamming.)

4. It ain’t so much the things we don’t know that get us into trouble. It’s the things we do know that just ain’t so. (Artemus Ward.)

5. In theory, theory and praxis are the same. In praxis, they aren’t. (Richard Nixon.)

6. Don’t worry about your difficulties with mathematics. I can assure you that mine are still greater. (Albert Einstein, to his neighbor’s young daughter.)

7. A Vulgar Mechanick can practice what he has been taught or seen done, but if he is in an error he knows not how to find it out and correct it, and if you put him out of his road, he is at a stand; Whereas he that is able to reason nimbly and judiciously about figure, force and motion, is never at rest till he gets over every rub. (Isaac Newton)
The Subject

8. Linear Algebra. The algebra of linear functions between finite dimensional vector spaces.

9. “Matrix” is a synonym for “linear function”

10. Matrix multiplication = function composition.

11. Norms. A vector norm is a function \( \| \| \) that associates a number with a vector such that the following properties hold:
\[
\begin{align*}
\|x\| & \geq 0 \\
\|x\| = 0 & \implies x = 0 \\
\|kx\| & = |k| \|x\| \\
\|x + y\| & \leq \|x\| + \|y\|
\end{align*}
\]
We saw many examples of norms such as \( \|x\|_1, \|x\|_2, \|x\|_\infty \) and, in general, \( \|x\|_p \).

12. Matrix Norms. Given a vector norm, the induced matrix norm is defined by
\[
\|A\| = \max_{\|x\|=1} \|Ax\|. 
\]
For example,
\[
\|A\|_1 = \max_j \sum_i |a_{ij}|, \quad \|A\|_\infty = \max_i \sum_j |a_{ij}|, \quad \text{and} \quad \|A\|_2 = \sqrt{\rho(A^T A)}.
\]

13. Orthogonal Matrices. \((A^{-1} = A^T)\) Columns form an orthonormal basis. Orthogonal matrices don’t amplify errors.

14. Linear System. \(Ax = b\)

15. \(A = LU\) + forward and backward substitution

16. We might just as well use a UL factorization, the choice of \(A = LU\) is strictly conventional.

17. Never invert a matrix! It’s unnecessary, it’s expensive, it’s likely to introduce additional round-off errors, and it destroys sparsity.

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18. Backward Error Analysis, condition number

\[ A(x - e) = b - r \]  
(4)

\[
\frac{1}{\|A\| \|A^{-1}\|} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|}\leq \|A\| \|A^{-1}\| \frac{\|r\|}{\|b\|} 
\]  
(5)

19. loose approximately \(\log_{10} \|A\| \|A^{-1}\|\) significant digits.

20. Special Case: A symmetric matrix \(A\) is positive definite if \(x^T Ax > 0\) for all \(x \neq 0\). Positive definite systems arise in minimization problems. Use the Cholesky decomposition: \(A = LL^T\). This does not require pivoting.

21. Least Squares: \(\|Ax - b\|_2 = \min\).
   - Normal equations \(A^T Ax = A^T b\). Conceptually simple but numerically inferior.
   - QR factorization approach is better.

22. Ill conditioning arises naturally and is incurable once it’s there. Use a clever basis to avoid it.

23. QR factorization. Compute via Householder Reflections

\[ H = I - 2uu^T, \quad \|u\|_2 = 1 \]  
(6)

24. Eigenvalue Problems \(Ax = \lambda x\). This is a nonlinear system. eigenvalues, eigenvectors, eigenvectors only determined up to a constant.

25. Eigenvalues are roots of the characteristic polynomial

\[ p(\lambda) = \det(A - \lambda I) \]  
(7)

26. For any polynomial \(p\) there is a matrix that has \(p\) as its characteristic polynomial, e.g., the companion matrix.

27. Gershgorin Theorem: Suppose \(Ax = \lambda x\). Then, for some \(i\),

\[ |a_{ii} - \lambda_i| \leq \sum_{j \neq i} |a_{ij}|. \]  
(8)

28. To find roots of a polynomial find the eigenvalues of its companion matrix.
29. **Power Method**: find one eigenvalue and corresponding eigenvector. Shift of Origin and inverse iteration.

30. **QR algorithm**: find all eigenvalues (and corresponding eigenvectors). Key ingredients include: simultaneous orthogonal iteration, use of orthogonal matrices, Hessenberg matrices, maintaining $O(n^2)$ effort at all times, decoupling, shifts of origin, executing complex double shifts in real arithmetic. The most complex algorithm we encountered this semester.

31. **Singular Value Decomposition.** For a number of linear algebra problems, if all else fails, the singular value decomposition (SVD) is your last resort. The singular value decomposition of an $m \times n$ matrix $A$ is given by

$$A = U\Sigma V^T$$

where $U$ is $m \times m$ orthogonal, $\Sigma$ is $m \times n$ diagonal, and $V$ is $n \times n$ orthogonal. Assuming $m \geq n$ the diagonal entries of $\Sigma$

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$$

are the singular values of $A$. The columns of $U$ and $V$ are the left and right singular vectors of $A$, respectively. The basic idea of using the SVD is to reduce and problem involving $A$ to an equivalent problem involving a diagonal matrix, using orthogonal matrices for the transformation. We proved that the SVD exists and discussed several applications, including

31.1 rank determination

31.2 computation of determinants, norms, and condition numbers

31.3 Linear Systems

31.4 Least Squares problems

31.5 data compression.

32. **Fixed Point Iteration.** Find a function $g$ such that $f(x) = 0$ is equivalent to $x = g(x)$. Then start with some $x_0$ and iterate:

$$x_{k+1} = g(x_k) \quad k = 0, 1, 2, \ldots$$
33. **Newton’s Method.** A special case of a fixed point iteration. Basic idea: *Linearize and Iterate.* That idea has a wide range of applications. For \( f(x) = 0 \) it becomes
\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}
\] (12)

34. The choice of **starting value** depends on the problem.

35. The fixed point iteration (11) converges of order \( p \) to \( \alpha \) if
\[
g(\alpha) = \alpha, \quad g'(\alpha) = g''(\alpha) = \ldots = g^{(p-1)}(\alpha) = 0, \quad g^{(p)}(\alpha) \neq 0.
\] (13)

36. **Aitken Acceleration.** Make a linearly convergent sequence \( p_0, p_1, p_2, \ldots \) into a quadratically convergent sequence \( \hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots \)
\[
\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}, \quad n = 0, 1, 2, \ldots \] (14)

37. **Polynomials.** Evaluate by Horner’s Scheme (nested multiplication, synthetic division) and its generalizations. The space of polynomials is closed under multiplication, addition, composition, integration, and differentiation. Degree, exact degree, degree no larger than, constant, linear, quadratic, cubic, quartic, quintic.

38. **Interpolation** means exact reproduction of data.

39. **Lagrange interpolation** = interpolation to function values by a polynomial.

39.1 **power form,** Vandermonde Matrix, Existence, Uniqueness

39.2 **Newton Form.** Add one point at a time.

39.3 **Lagrange Form.** A special case of cardinal form.

40. **Interpolation Error:** Let \( p \) be the polynomial interpolating to \( f \) at \( n+1 \) distinct points \( x_0, \ldots, x_n \). Then
\[
f(x) - p(x) = \frac{\prod_{i=0}^{n} (x-x_i)}{(n+1)!} f^{(n+1)}(\xi).
\] (15)
41. **The Runge Phenomenon.** Interpolation by a high degree polynomial can lead to large oscillations (and errors).

42. **Hermite interpolation** = interpolation to function and derivative values by a polynomial.

43. **Cardinal Form of an Interpolant** Use the function (and derivative) values as coefficients.

44. **Numerical Integration** Integrate the interpolant.

45. **Newton Cotes Formulas** Open, closed, simple, composite. Interpolate at equally spaced points.

46. **Gaussian Quadrature.** Use weights and knots to obtain exactness for polynomials of degree as high as possible. Gaussian genius allowed solution of the nonlinear system.

47. **Numerical Differentiation.** It’s difficult or impossible. However, if it can’t be helped differentiate the interpolant. Choose the knots judiciously.

48. **Symmetry.** Abandon it only if you have to.

49. **Method of Undetermined Coefficients.** Write down the general form of a method. Find the coefficients by requiring exactness for specific functions.

50. **Piecewise Polynomial Functions.**
   
   50.1 **Piecewise linear:** Simple, but effective, particularly in Computer Graphics which is ultimately pixel based.

   50.2 **Piecewise Cubic Hermite:** Once differentiable, interpolate to function values and derivatives.

   50.3 **Cubic Splines:** twice differentiable, interpolate to function values only. Two end conditions must be imposed: forced end condition, natural end condition, not-a-knot condition.

51. **Approximation of Functions.** Minimize the norm of the difference of a function and a linear combination of basis functions. The norm can be the 2 norm, or some other norm.

52. **Basis Functions.** Pick them according to your problem. Use them to incorporate singularities (explosions), exponential growth or decay, or periodicity, for example.
53. **Orthonormal Basis.** Many kinds of orthogonal polynomials. They depend on your choice of the inner product.

54. **Discretization.** Computers are finite and solutions of differential equations consists of infinitely many points. The first step of solving a DE numerically consists of reducing the problem to one of finite dimension.

55. **Truncation Error** = true solution minus approximate solution. You are truncating some Taylor series. Other errors, such as round-off errors, may be significant, but are usually ignored.

56. **Global Error** = error in the actual solution.

57. **Consistency** means the local error is of a sufficiently small order.

58. **Stability** means that errors, once introduced, do not get amplified. Amplification of errors means multiplication with a factor whose absolute value exceeds 1.

59. **Convergence.** A method is convergent if the error can be made arbitrarily small by expending enough effort. If a method is convergent then you are not limited by the method itself as far as the attainable accuracy is concerned.

60. **Grand Theme** in theoretical numerical analysis

\[ \text{Convergence } \iff \text{Consistency and Stability} \] (16)

61. **ODE-IVPs**

\[ y' = f(x, y), \quad y(a) = y_0. \] (17)

Technical assumptions (e.g., Lipschitz continuity) ensure existence and uniqueness of the method. Higher order DEs can be reduced to first order systems, by giving names to intermediate derivatives. If useful we may assume the system is autonomous, \( y' = f(y) \).

62. **Stiff ODEs** The problem (17) is stiff if for an explicit method the step-size is limited by stability rather than by local accuracy. Other definitions of stiffness include: the
eigenvalues of $f_y$ are widely scattered in the left half plane, neighboring solutions converge rapidly to the analytical solution, or the solution has components with widely disparate time scales.

63. Discretization

$$x_n = a + nh, \quad y_n \approx y(x_n), \quad y_n = ?, \quad f_n = f(x_n, y_n).$$ \hfill (18)

64. Linear Multistep Methods, LMM

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$ \hfill (19)

65. $k$ is the step number, $\alpha_k = 1$, if $\beta_k = 0$ the method is explicit, otherwise it is implicit, if $k > 1$ the method is incomplete without a starting strategy.

66. Convergence of a linear multistep method means that

$$\lim_{h \to 0, x = a + nh} y_n = y(x)$$ \hfill (20)

for all $x$ and starting strategies $y_\mu = n_\mu(h)$, $\mu = 0, \ldots, k - 1$ such that $\lim n_\mu(h) = y_0$.

67. For the method to be convergent for all IVPs (17) it must be convergent when $y'_0 = 0$ and when $y'_0 = 1$. Just considering those two problems led to a number of necessary conditions.

68. Characteristic Polynomials The first and second characteristic polynomials of a LMM are given by

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j \quad \text{and} \quad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j,$$ \hfill (21)

respectively.

69. Linear Difference Equations are of the form

$$\sum_{j=0}^{k} \gamma_j y_{n+j} = z_n$$ \hfill (22)
If \( z_n = 0 \) the equation is **homogeneous**. The general solution of a homogeneous linear difference equation is a linear combination of powers of roots of its characteristic polynomials, with adjustments in case there are multiple roots.

### 70. The general solution of a linear problem

is any particular solution plus the general solution of the homogeneous problem.

### 71. For linear problems, the global truncation error satisfies

the same equation as the solution, except the right hand side is different (and equals the local truncation error).

### 72. The method (19) is **zero-stable** if

\[
\rho(\xi) = 0 \implies |\xi| \leq 1 \quad \text{and} \quad \rho(\xi) = \rho'(\xi) = 0 \implies |\xi| < 1.
\]

### 73. The method (19) is **consistent** if

\[
\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1).
\]

### 74. For the method (19) to converge when \( y' = 0 \) and \( y' = 1 \) it must be consistent and zero-stable.

### 75. If the method (19) is consistent and zero-stable it will converge for all problems (17).

### 76. Most Amazing!

LMM convergent for all \( y' = f(x, y) \) \iff LMM convergent for \( y' = 0 \) and \( y' = 1 \)

### 77. The **Order** of a LMM is the defined in terms of the local truncation error:

\[
LTE = \sum_{j=0}^{k} \alpha_j y(x_{n+j}) - h \sum_{j=0}^{k} \beta_j y'(x_{n+j}) = C_{p+1} h^{p+1} y^{(p+1)}(x_n) + H.O.T.
\]

\( p \) is the order, and \( C_{p+1} \) is the error constant.

### 78. A LMM is consistent if and only if its order is at least 1.

### 79. Test Equation.

Compare the general solution of the DE with the general solution of the discretization, and
compare qualitative features. Above, we use $y' = 0$ and $y' = 1$ as test equations.

80. Next test equation obtained from $y' = f(x, y)$ by linearizing, freezing, homogenizing, and diagonalizing.

$$y' = \lambda y$$

(27)

where $\lambda$ is complex, and plays the role of an eigenvalue of the Jacobian of $f$.

81. **Absolute Stability**

LMM absolutely stable for $h\lambda \iff$ all solutions of $y' = \lambda y \to 0$ as $n \to \infty$

$$\iff \pi(h\lambda, \xi) = \rho(\xi) - h\lambda\sigma(\xi) = 0 \implies |\xi| < 1.$$  

(28)

82. The **region of absolute stability** of an LMM is the set of all complex numbers $h\lambda$ for which the LMM is absolutely stable. The ideal region of absolute stability is the open left half plane.

83. A method is **A-stable** if its region of absolute stability includes the open left half plane.

84. An explicit method cannot have an infinite region of absolute stability.

85. The order of an A-stable LMM cannot exceed 2.

86. **Adams Methods**

$$y_{n+k} - y_{n+k-1} = h \sum_{j=0}^{k-1} \beta_j f_{n+j} \quad p = k \quad \text{Adams-Bashforth}$$

$$y_{n+k} - y_{n+k-1} = h \sum_{j=0}^{k} \beta_j f_{n+j} \quad p = k + 1 \quad \text{Adams-Moulton}$$

(29)

87. **Backward Differentiation Methods, BDF**

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \beta_k f_{n+k}, \quad k < 7, \quad p = k$$

(30)

BDF methods have infinite regions of absolute stability but are not zero-stable if $k > 6$. BDF methods are widely used for stiff problems.
88. For Adams methods, the roots of the stability polynomial have the right behavior close to the origin, for BDF methods, they have the right behavior as $|h\lambda|$ goes to infinity.

89. For LMMs, the order of the global truncation errors is one less than that of the local truncation error.

90. **Predictor Corrector Methods**: Predict, Evaluate, Correct. PECE, P(EC)²E. Predictor is explicit, corrector implicit, PC method is explicit.

91. **Essence of Adaptive Methods**: Use an error estimate. Based on the difference of two approximations.

92. **Milne’s device**. Use the difference of predicted and corrected value.

93. **Variable Step Variable Order Methods, VSVO**. Six ingredients:

93.1 1. A family of methods, e.g., Adams-Bashforth-Moulton PECE, $k = 1, \ldots, 13$.

93.2 2. A starting strategy. Start with a one step method and a small step.

93.3 3. A local error estimator. e.g., Milne’s device.

93.4 4. A Step-size/order policy.

93.5 5. A technique to change the step-size.

93.6 6. A technique to change the order.

94. **ODE-BVPS** Considered three methods:

95. **Shooting**. Reduce the BVP to an IVP.

96. **Finite Differences**. Replace derivatives with differences and solve the resulting equations. The orders of local and global truncation errors are the same.

97. **Ritz Method**. Minimize over a finite dimensional subspace. Build boundary conditions into the subspace.

98. **Finite Elements**. A special case of the Ritz Method. Use functions with small support and have nodal values as parameters. The “elements” are functions spanning the approximating space, or pieces of the domain. Variations are the **Galerkin Method** and **Collocation**.

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99. **Local Truncation Error.** Write the method as something equals zero, substitute exact values, and expand into a Taylor series.

100. **Future Classes:**

100.1 **5610-20** Covers the same material as our class, but in more depth.

100.2 **6610-20** Covers the same material as our class on the graduate level.

100.3 Both 5620 and 6620 cover numerical DEs, 5610 and 6610 cover everything else.

100.4 **6630** Advanced Numerical PDEs. (5620 and 6620 both cover introductory PDEs.)

100.5 You may be ready for 6610-20. If you are interested I recommend you talk with the instructor (Akil Narayan) and perhaps attend some of his first lectures before you make a decision.

101. **Linear Programming**

102. **Simplex Method**

103. **Method of Lines** \( u_t = u_{xx} \)

104. **Crank-Nicolson**

105. **von Neumann Stability**

106. **Douglas**