The bulk of this summary comprises the slightly edited summaries for the three past midterm exams. In addition, at the end there is a summary of Laplace Transforms.

The final exam will take place on Friday, August 5, 2016, 10:00-12:00 noon, in our regular classroom. Its format will be like that of the midterms. The exam will cover the whole semester, i.e., chapters 1–7.

The following list of specific topics in this review is neither complete nor self contained. Some items brush over technical details like required degrees of smoothness. Each point listed here should trigger your memory of many connected facts and concepts.

1. Recognize, and know how to solve DEs and IVPs of the form

1.1 \( \frac{dy}{dx} = f(x) \) (integration problems)

1.2 \( \frac{dy}{dx} = f(x)g(y) \) (separable DEs)

1.3 \( \frac{dy}{dx} + P(x)y = Q(x) \) (first order linear)

1.4 \( M(x, y)dx + N(x, y)dy = 0 \) where \( M_y = N_x \) (exact equation)

1.5 second order equations where \( x, y, \) or \( y' \) are missing. In all cases use the substitution \( p(x) = y' \).

2. Know how to draw and interpret slope (or direction) fields.

3. autonomous equations, \( y' = f(y) \), their equilibrium solutions, and the stability, or lack of stability, of those solutions.

4. position - velocity - acceleration

5. constant and non-constant acceleration

\[ h'' = v' = a = -g, \quad h'' = -ky - g, \quad h'' = -ky^2 - g \]  

6. Exponential growth and decay

\[ y' = ky \]

7. Logistic growth

\[ y' = ky(M - y) \]

8. Doomsday/Extinction equation

\[ y' = ky(y - M) \]

9. Mixture problems

10. Newton’s Law of Cooling
11. While we did discuss (briefly!) numerical methods, there will be no questions from sections 2.4-6 on the exam.

12. An operator $L$ is **linear** if
\[ L(u + v) = Lu + Lv \quad \text{and} \quad L(ku) = kLu \quad (5) \]
for all $u$ and $v$ in its domain, and real scalars $k$.

13. The general solution of a linear problem
\[ Lx = f \quad (6) \]
is of the form
\[ x = x_p + x_c \quad (7) \]
where $x_p$ is *any particular solution* of (7) and $x_c$, the **complementary solution**, is *the general solution* of the homogeneous problem
\[ Lx = 0. \quad (8) \]
Thus the problem (7) is **homogeneous** if $f = 0$.

14. The concept of **linear independence** of functions $f_1, \ldots, f_n$ is identical to that of the linear independence of vectors: The set \{ $f_1, f_2, \ldots, f_n$ \} is **linearly independent** if
\[
\sum_{k=1}^{n} \alpha_k f_k = 0 \quad \implies \quad \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0. \quad (9)
\]

15. We saw that the (sufficiently often differentiable) functions $f_1, f_2, \ldots, f_n$ are linearly independent if the Wronskian
\[
W = \begin{vmatrix}
  f_1 & f_2 & f_3 & \cdots & f_n \\
  f'_1 & f'_2 & f'_3 & \cdots & f'_n \\
  f''_1 & f''_2 & f''_3 & \cdots & f''_n \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  f^{(n-1)}_1 & f^{(n-1)}_2 & f^{(n-1)}_3 & \cdots & f^{(n-1)}_n
\end{vmatrix} \neq 0. \quad (10)
\]
In this notation, the vertical bars indicate a determinant.

16. We considered in particular *$n$-th order constant coefficient linear* differential equations of the form
\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x) \quad (11) \]
where the coefficients $a_0, \ldots, a_n$ are real constants (and $a_n \neq 0$).

17. To find the complementary solution $y_c(x)$, i.e, the solution of
\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \quad (12) \]
we solve the **characteristic equation**
\[ p(r) = 0 \quad (13) \]
where $p$ is the **characteristic polynomial**
\[ p(r) = a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0. \quad (14) \]

18. Once we know the roots of $p$ we can construct $n$ linearly independent solutions of (12) using the following observations:

18.1 if $r$ is a real root of $p$ then $y(x) = e^{rx}$ is a solution of (12).
18.2 If \( r = a + bi \) is a complex root of \( b \) then its conjugate is also a root and the two functions
\[
y_1 = e^{ax} \cos bx \quad \text{and} \quad y_2 = e^{ax} \sin bx
\]
are two solutions of (12).

18.3 If \( r \) is a (real or complex) root of multiplicity \( m > 0 \), i.e.,
\[
p(r) = p'(r) = \ldots = p^{(m-1)}(r) = 0, \quad p^{(m)}(r) \neq 0
\]
then we can construct additional solutions of (12) by multiplying the solutions obtained by the preceding rules with \( x, x^2, \ldots, x^{m-1} \).

18.4 The general solution of (12) is a linear combination of the form
\[
y_p(x) = \sum_{k=1}^{n} \alpha_k y_k(x)
\]
where the \( y_k \) are \( n \) linearly independent solutions constructed by the points above.

18.5 To find a particular solution of (11) we considered two cases and two major methods:

18.6 The **Method of Undetermined Coefficients**. It is applicable when the right hand side \( f \) and *all of its derivatives* can be written as a linear combination of finitely many linearly independent functions. Examples for such functions \( f \) include polynomials, exponentials, and linear combinations of \( \sin \) and \( \cos \) functions. Suppose those basis functions do not solve the homogeneous problem (12). Then let \( y_p \) be a linear combination of those basis functions with undetermined but constant coefficients, substitute in (11), collect terms, and match them with the expansion of \( f \) as a linear combination of those basis functions. You obtain a linear system of equations for the coefficients. If some of the basis functions do satisfy the homogeneous problem then multiply them with the least power of \( x \) that makes them no longer satisfy the homogeneous equation.

18.7 In the method of **variation of parameters** write the particular solution as a linear combination
\[
y_p(x) = \sum_{k=1}^{n} u_k(x)y_k(x)
\]
where the linearly independent functions \( y_k \) satisfy the homogeneous problem (12) and the coefficients \( u_k(x) \) are functions of \( x \) rather than constants. The coefficients satisfy the linear system
\[
\begin{bmatrix}
y_1 & y_2 & y_3 & \ldots & y_n \\
y_1' & y_2' & y_3' & \ldots & y_n' \\
y_1'' & y_2'' & y_3'' & \ldots & y_n'' \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \ldots & y_n^{(n-1)}
\end{bmatrix}
\begin{bmatrix}
u_1' \\
u_2' \\
u_3' \\
\vdots \\
u_n'
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
f(x)
\end{bmatrix}
\]

Solve it and find the coefficients by integration.

19. We learned how to convert a linear combination of \( \sin \) and \( \cos \) functions into a single \( \cos \) function:
\[
A \cos \omega t + B \sin \omega t = C \cos(\omega t - \alpha)
\]
where
\[
C = \sqrt{A^2 + B^2} \quad \text{is the amplitude,}
\]
\( \omega \) is the **circular frequency**, \( \frac{\omega}{2\pi} \) is the **frequency**, and \( \nu = \frac{1}{\omega} \) is the **period**, \( \alpha = \begin{cases} \tan^{-1}(B/A) & \text{if } A > 0 \text{ and } B > 0 \\ \pi + \tan^{-1}(B/A) & \text{if } A < 0 \text{ and } B > 0 \\ 2\pi + \tan^{-1}(B/A) & \text{if } A > 0 \text{ and } B < 0 \end{cases} \) is the **phase angle**.
20. A simple model of mechanical vibrations is the second order constant coefficient linear differential equation

\[ mx'' + cx' + kx = F(t). \] (22)

Here, \( x(t) \) is the displacement from equilibrium at time \( t \), \( m \) is the mass of the vibrating object, \( c \) is the damping constant, \( k \) is the spring constant, and \( F(t) \) is the external force acting on the system. We saw several applications, and we know how to solve this differential equation.

21. In the homogeneous problem,

\[ mx'' + cx' + kx = 0 \] (23)

there are three possible cases, corresponding to the number of real solutions of the characteristic equation:

21.1 Overdamped, \( c^2 > 4km \), 2 real solutions of \( p(r) = 0 \), \( x(t) \) is a linear combination of two decaying exponentials, \( x \) crosses the \( t \)-axis at most once.

21.2 Critically Damped, \( c^2 = 4km \), 1 real solution of \( p(r) = 0 \), \( x(t) \) is a linear combination of \( e^{rt} \) and \( te^{rt} \) where \( r < 0 \). \( x \) crosses the \( t \)-axis at most once, and the graphs of \( x(t) \) look much like the graphs of the overdamped solutions.

21.3 Underdamped, \( c^2 < 4km \), 2 conjugate complex solutions of \( p(r) = 0 \), \( x(t) \) is an oscillation multiplied with a decaying exponential. The graph of \( x(t) \) crosses the \( t \)-axis infinitely often.

22. The simple pendulum gives rise to the equation

\[ \theta'' + \frac{g}{L}\sin \theta = 0 \] (24)

which is of the form (22). The solution of the DE is an undamped harmonic motion. Our derivation illustrated two important techniques. The first was to use the principle of conservation of energy. The other, more minor, but frequently used, idea was to approximate the sine of an angle by the angle itself:

\[ \sin \theta \approx \theta \] (25)

which is reasonable for small angles \( \theta \).

23. In the inhomogeneous case we restricted our attention to simple harmonic external forces:

\[ mx'' + cx' + kx = F_0 \cos(\omega t). \] (26)

We discussed the phenomenon of resonance. This occurs when \( c = 0 \) and \( \omega \) equals the natural circular frequency of the system. In that case the amplitude grows linearly and without bound. In the presence of damping, \( c > 0 \), there is a circular frequency less than the natural frequency, for which the amplitude of the solution is maximized. This is called practical resonance.

24. We briefly analyzed electric circuits which give rise to very similar differential equations as those discussed for mechanical vibrations.

25. A major class of problems are boundary value problems where a specific solution of a differential equation is defined by conditions at more than one value of the independent variable. A typical problem is

\[ x'' = f(t, x, x'), \quad y(a) = A \text{ and } y(b) = B. \] (27)

26. Boundary value problems are very different from initial value problems. An initial value problem usually has a unique solution under reasonable smoothness assumptions. A boundary value problem may have none, one, or infinitely many solutions. Here are some examples:

\[ x'' = -x, \quad \begin{cases} x(0) = A, x(1) = B \quad \text{one unique solution} \\ x(0) = 0, x(\pi) = 1 \quad \text{no solution} \\ x(0) = 0, x(\pi) = 0 \quad \text{infinitely many solutions} \end{cases} \] (28)
27. An interesting and important special case of infinitely many solutions are eigenfunctions. Consider a linear differential operator \( \Lambda \) with (suitably many) homogeneous boundary conditions. \( x \) is an eigenfunction of this problem with corresponding eigenvalue \( \lambda \) if \( x \neq 0 \) and

\[
\Lambda x = \lambda x \quad \text{and} \quad x \text{ satisfies the BCs.}
\]  

(29)

For example, suppose that

\[
\Lambda x = -x'', \quad x(0) = x(L) = 0.
\]  

(30)

The eigenfunctions and corresponding eigenvalues of this problem are

\[
x_k = \sin \frac{k\pi t}{L} \quad \text{and} \quad \lambda = -\left(\frac{k\pi}{L}\right)^2
\]  

(31)

where \( k = 1, 2, 3, 4, \ldots \). If \( x \) is an eigenfunction then any non-zero multiple of it is also an eigenfunction, with the same eigenvalue. The zero function is not an eigenfunction.

28. Chapter 4 deals with Systems of Differential Equations. The phrase simply means that we have more than one equation, i.e., we have several dependent variables (but only one independent variable for an ODE).

29. Any system of high order differential equations can be converted to a first order system of differential equations. The basic technique consists of just giving names to the derivatives, adding the definitions of those names to the system, and rewriting the original equations in terms of the new variables. For example, the second order equation

\[
x'' = f(t, x, x')
\]

is equivalent to the first order system

\[
x' = y \quad \text{and} \quad y' = f(t, x, y)
\]  

(32)

30. It is sometimes possible, and occasionally useful, to convert a system of differential equations to a single high order equation. We know in particular how to do this if the differential equations are constant coefficient linear. (The textbook refers to the associated differential operators as polynomial operators.) A crucial fact about those operators is that they commute. As a consequence we can use a variation of Cramer’s Rule. For example, consider the system

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{13} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}.
\]  

(33)

The dependent variable \( x_1 \) satisfies the single high order equation

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{13} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\]  

(34)

where the vertical bars indicate determinants. The other variables \( x_2 \) and \( x_3 \) satisfy analogous equations. The \( L_{ij} \) are polynomial operators of the form

\[
Lx = \sum_{k=0}^{n} a_k \frac{d^k}{dt^k} x
\]  

with constant coefficients \( a_k \). The conversion works more generally so long as the \( L_{ij} \) all commute.

31. A system of differential equations is simply a set of several differential equations. In this course we only consider ordinary DEs, so there is only one independent variable, but there may be several dependent variables.

32. Any system of DEs can be converted to a first order system.
33. A system is **autonomous** if the derivatives do not depend explicitly on the independent variable.

34. A first order system can be converted to a high order system by **elimination** but the opposite process of converting a high order equation to a first order system is more common.

35. We considered linear systems of the form
   \[ x' = Ax + f(t) \]  
   (36)
   where \( x \) and \( f \) are vectors in \( \mathbb{R}^n \) and \( A \) is an \( n \times n \) matrix. Its entries are usually constant, although the textbook does consider functions of \( t \) in some places.

36. The system (36) is **homogeneous** if \( f = 0 \) and **inhomogeneous** otherwise. Thus the homogeneous problem is
   \[ x' = Ax. \]  
   (37)

37. If \( Av = \lambda v \), i.e., \( v \neq 0 \) is an eigenvector of \( A \) with corresponding eigenvalue \( \lambda \) then
   \[ x(t) = e^{\lambda t}v \]  
   (38)
is a solution of (37).

38. If \( A \) has \( n \) linearly independent eigenvectors \( v_i \) with corresponding real eigenvalues \( \lambda_i \) then the general solution of the homogeneous problem can be written as
   \[ x(t) = \sum_{i} e^{\lambda_i t}v_i. \]  
   (39)

39. If \( \lambda = p + qi \) is a complex eigenvalue with corresponding eigenvector \( v = a + bi \) (where \( i^2 = -1 \)) then two linearly independent real solutions of the homogeneous problem (37) are
   \[ x_1(t) = e^{pt} (a \cos qt - b \sin qt) \]
   \[ \text{and} \quad x_1(t) = e^{pt} (b \cos qt + a \sin qt) \]  
   (40)

40. An eigenvalue \( \lambda \) of a matrix \( A \) has **algebraic multiplicity** \( a \) if it is a root of multiplicity \( a \) of the characteristic polynomial
   \[ p(\lambda) = \det(A - \lambda I). \]  
   (41)
   \( \lambda \) has **geometric multiplicity** \( g \) if the dimension of the linear space spanned by the eigenvectors corresponding to \( \lambda \) is \( g \). \( \lambda \) is **complete** if \( a = g \), and **defective** otherwise. The matrix \( A \) is **defective** if the dimension of the space spanned by its eigenvectors is less than \( n \), and **non-defective** otherwise. A defective matrix has at least one defective eigenvector.

41. The following table lists examples of various kinds of matrices:

<table>
<thead>
<tr>
<th>Eigenvectors</th>
<th>Singular</th>
<th>Non-Singular</th>
</tr>
</thead>
</table>
| Defective    | \[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\] | (42) |
| Non-defective| \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] |

42. We did not discuss this in class, but as a quick reminder: symmetric matrices have real eigenvalues, and their eigenvectors can be chosen so as to be orthogonal.
43. For defective eigenvalues we introduced the notion of **generalized eigenvectors**. Suppose \( \lambda \) is an eigenvalue with algebraic multiplicity \( k \) and algebraic multiplicity 1. Suppose also that \( v_1 \) is an eigenvector corresponding to \( \lambda \). Then \( v_1, v_2, \ldots, v_k \) defined by

\[
\begin{align*}
A v_1 &= \lambda v_1 \\
(A - \lambda I) v_2 &= v_1 \\
(A - \lambda I) v_3 &= v_2 \\
&\vdots \\
(A - \lambda I) v_k &= v_{k-1}
\end{align*}
\] (43)

are generalized eigenvectors of \( A \).

44. If \( \lambda \) is an eigenvalue of \( A \) with geometric multiplicity \( k \), and \( v_1, v_2, \ldots, v_k \) is a chain of associated generalized eigenvectors then the following functions are \( k \) linearly independent corresponding solutions:

\[
\begin{align*}
x_1 &= e^{\lambda t} v_1 \\
x_2 &= e^{\lambda t} (v_1 t + v_2) \\
x_3 &= e^{\lambda t} \left( \frac{1}{2} v_1 t^2 + v_2 t + v_3 \right) \\
&\vdots \\
x_k &= e^{\lambda t} \left( \frac{1}{(k-1)!} v_1 t^{k-1} + \ldots + v_{k-1} t + v_k \right)
\end{align*}
\] (44)

Notice the pattern: each factor multiplying the exponential is obtained by integrating the preceding factor, and using the new generalized eigenvector as the integration constant.

45. A **fundamental matrix** \( \Phi(t) \) of the homogeneous system (37) is a non-singular matrix whose columns satisfy the differential equation (37). Any fundamental matrix also satisfies the differential equation

\[
\Phi'(t) = A \Phi(t).
\] (45)

46. The solution of the (constant coefficient linear homogeneous) initial value problem

\[
x' = Ax, \quad x(0) = x_0
\] (46)

can be written in terms of a fundamental matrix as

\[
x(t) = \Phi(t) \Phi(0)^{-1} x_0.
\] (47)

47. The **Matrix Exponential** (and many other matrix functions) can be defined in terms of power series. Specifically,

\[
e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}
\] (48)

48. In terms of a fundamental matrix \( \Phi \) the matrix exponential can be computed as

\[
e^{At} = \Phi(t) \Phi(0)^{-1}.
\] (49)

49. In terms of the matrix exponential the unique solution of the initial value problem (46) can be written as

\[
x(t) = e^{At} x_0.
\] (50)
50. As for any linear problem, the solutions of the homogeneous problem (37) form a linear space (of dimension \( n \)) and the general solution of the problem (36) can be written as any particular solution at all plus the general solution of the homogeneous problem:

\[
x(t) = x_c(t) + x_p(t).
\]  

(51)

51. One method to find a particular solution is the method of undetermined coefficients. It works the same way as in the linear constant coefficient high order single equation case: write the particular solution as a linear combination of suitable terms, substitute in the original equation, match terms, and compute the coefficients.

52. The method of Variation of Parameters gives rise to the formula

\[
x_p(t) = \Phi(t) \int \Phi(t)^{-1} f(t) dt
\]

(52)

where \( \Phi(t) \) is a fundamental matrix for the homogeneous problem (37). The general solution of the inhomogeneous problem (36) can then be written as

\[
x(t) = \Phi(t)c + \Phi(t) \int \Phi(t)^{-1} f(t) dt.
\]

(53)

53. The solution of the initial value problem (46) can be written variously as

\[
x(t) = \Phi(t) \Phi(t)^{-1} x_0 + \Phi(t) \int_0^t \Phi(s)^{-1} f(s) ds
\]

\[
e^{A_t} x_0 + \int_0^t e^{-A(s-t)} f(s) ds
\]

(54)

54. Chapter 6 illustrates nonlinear systems, particularly the autonomous system of two equations

\[
\begin{align*}
x' &= F(x, y) \\
y' &= G(x, y)
\end{align*}
\]

(55)

The solutions of this equation can be thought of as parameterizing curves in the \( x - y \) plane, which in this context is referred to as the phase plane. Those curves are trajectories (or solution curves) of the system. If the trajectories are closed they are also called orbits.

55. The fundamental idea of analyzing nonlinear systems is to approximate them locally by linear systems.

56. An equilibrium point of the nonlinear system (55) is a point \((x_0, y_0)\) for which

\[
x' = y' = 0.
\]

(56)

57. Suppose

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

(57)

is non-singular. Then the unique equilibrium point of the linear system

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

(58)
is the origin.

58. The significance of assuming that the matrix in the linearization (58) is non-singular is that the equilibrium point is isolated, i.e., there is a neighborhood of it that contains no other equilibrium points.

59. The behavior of the solutions of the system (58) depend on the eigenvalues of $A$. We consider the case that $t$ goes to infinity. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $A$, and $v_1$ and $v_2$ the corresponding eigenvectors.

60. Consider first the case that the eigenvalues are real. The following subcases are possible.

60.1 $\lambda_1 < \lambda_2 < 0$. The origin is an asymptotically stable improper nodal point. Incoming trajectories are parallel to $v_2$ at the origin.

60.2 $\lambda_1 > \lambda_2 > 0$. The origin is an unstable improper nodal point. Outgoing trajectories are parallel to $v_2$ at the origin.

60.3 $\lambda_2 > 0 > \lambda_1$. The origin is an unstable saddle point.

60.4 $\lambda_1 = \lambda_2 < 0$, $v_1$ and $v_2$ are linearly independent. The origin is a stable proper nodal point, all incoming trajectories are rays.

60.5 $\lambda_1 = \lambda_2 < 0$, $\lambda_1$ is defective. The origin is a stable improper nodal point, all incoming trajectories are parallel to $v_1$.

60.6 $\lambda_1 = \lambda_2 > 0$, $v_1$ and $v_2$ are linearly independent. The origin is an unstable proper nodal point, all outgoing trajectories are rays.

60.7 $\lambda_1 = \lambda_2 > 0$, $A$ is defective. The origin is an unstable improper nodal point, all outgoing trajectories are parallel to $v_1$.

61. Suppose $\lambda_1$ and $\lambda_2$ are conjugate complex and $r$ is the real part of both. The following subcases are possible:

61.1 $r > 0$. The origin is an unstable spiral point.

61.2 $r < 0$. The origin is a stable spiral point.

61.3 $r = 0$. The origin is a stable equilibrium point. The trajectories are ellipses centered at the origin.

62. The linearization of the nonlinear problem (55) about the equilibrium point $(x_0, y_0)$ is the constant coefficient homogeneous linear system

$$
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
F_x(x_0, y_0) & F_y(x_0, y_0) \\
G_x(x_0, y_0) & G_y(x_0, y_0)
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}.
$$

(59)

63. Roughly speaking, sufficiently close to the equilibrium point the solutions of the nonlinear problem show the same qualitative behavior as the solution of its linearization.

64. There are some subtleties, however. For example, the origin forms a stable spiral point of the system

$$
\begin{align*}
x' &= y \\
y' &= -\sin x - y|y|
\end{align*}
$$

(60)

whereas the solutions of the linearized system

$$
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
x' \\
y'
\end{bmatrix}
$$

(61)

form closed orbits around the origin. As the solution of (60) approaches zero the damping effect approaches zero as well, but the spiral nature of the solutions persists. A more precise discussion is in section 6.2 of the textbook.
65. We considered various applications.

66. **Mixture Problems.** We have several interacting tanks. The concentration of each is one of the dependent variables.

67. **Predator/Prey:**

\[
\begin{align*}
x' &= ax - pxy = x(a - py) & a, p > 0 \\
y' &= -by + qxy = y(qx - b) & b, q > 0
\end{align*}
\]  

(62)

There are two equilibrium points, the origin and \( S = (b/q, a/p) \). The origin is an unstable saddle point, and \( S \) is stable. The trajectories in the first quadrant form closed orbits around \( S \).

68. **Competing Species**

\[
\begin{align*}
x' &= a_1 x - b_1 x^2 - c_1 xy = x(a_1 - b_1 x - c_1 y) & a_1, b_1, c_1 > 0 \\
y' &= a_2 y - b_2 y^2 - c_2 xy = y(a_1 - b_1 y - c_1 x) & a_2, b_2, c_2 > 0
\end{align*}
\]  

(63)

These were obtained from the logistic equation for each species by adding the competition terms \( c_1 xy \) and \( c_2 xy \). There are four equilibrium points corresponding to

68.1 The origin, no life.

68.2 \((0, a_2/b_2), (a_1/b_1, 0)\), only one species survives.

68.3 \( E = (x_E, y_E) \) where

\[
x_E = \frac{b_2 a_1 - c_1 a_2}{b_1 b_2 - c_1 c_2} \quad \text{and} \quad y_E = \frac{b_1 a_2 - c_2 a_1}{b_1 b_2 - c_1 c_2}
\]  

(64)

The two species coexist.

The coexistence point \( E \) is stable if \( c_1 c_2 < b_1 b_2 \) and unstable if \( c_1 c_2 > b_1 b_2 \).

69. **Undamped Nonlinear Springs.** We converted the nonlinear spring equation

\[
mx'' = -kx + \beta x^3
\]  

(65)

to the system

\[
\begin{align*}
x' &= y \\
y' &= -kx + \beta x^3
\end{align*}
\]  

(66)

and visualized solutions in the **Position-Velocity Phase Plane**, the \( x - y \) plane. The spring is **soft** if \( \beta > 0 \) and **hard** if \( \beta < 0 \).

70. For the hard spring the origin is the only equilibrium point, its trajectories are closed ovals.

71. For the soft spring there are two additional equilibrium points. Both are unstable saddle points.

72. **Damped Nonlinear Springs.** The addition of a damping term led to

\[
mx'' = -cx' - kx + \beta x^3.
\]  

(67)

The origin becomes a nodal or spiral sink.

73. **Damped Nonlinear Pendulum Oscillations** led to the system

\[
\begin{align*}
x' &= y \\
y' &= -\omega^2 \sin x - cy
\end{align*}
\]  

(68)

which in turn led to some pretty phase plane portraits.
74. We discussed chaos which is a rich subject recommended for further study.

75. The **Laplace Transform** of a function \( f \) is defined by

\[
\mathcal{L}\{f(t)\} = \mathcal{L}\{f\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)dt.
\]  

(69)

In general we use capital letters for the Laplace Transform and corresponding lower case letters for its preimage. The Laplace operator is invertible and we use the notation \( \mathcal{L}^{-1} \) for its inverse:

\[
F = \mathcal{L}\{f\} \iff f = \mathcal{L}^{-1}\{F\}.
\]

(70)

Throughout our discussions we used \( s \) as the independent variable in the Laplace Transform of a function and \( t \) as the independent variable in its preimage.

76. Table 1 gives a Table of frequently used formulas for Laplace Transforms.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>( \int_{0}^{\infty} e^{-st} f(t)dt )</td>
<td>monomials</td>
</tr>
<tr>
<td>( t^n )</td>
<td>( \frac{s^n}{n!} )</td>
<td>exponentials</td>
</tr>
<tr>
<td>( e^{at} )</td>
<td>( \frac{1}{s-a} )</td>
<td></td>
</tr>
<tr>
<td>( \sin \omega t )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
<td></td>
</tr>
<tr>
<td>( \cos \omega t )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
<td></td>
</tr>
<tr>
<td>( f^{(n)}(t) )</td>
<td>( s^n Lf - \sum_{k=0}^{n-1} s^k f^{n-1-k}(0) )</td>
<td>derivatives</td>
</tr>
<tr>
<td>( e^{at} f(t) )</td>
<td>( \mathcal{L}{f}(s-a) )</td>
<td>shift in ( s )</td>
</tr>
<tr>
<td>( u(t-a) = \begin{cases} 0 &amp; \text{if } t &lt; a \ 1 &amp; \text{if } t \geq a \end{cases} )</td>
<td>( \frac{e^{as}}{s} )</td>
<td>step function</td>
</tr>
<tr>
<td>( \mathcal{L}{u(t-a)f(t-a)} )</td>
<td>( e^{as} \mathcal{L}{f} )</td>
<td>shift in ( t )</td>
</tr>
<tr>
<td>( \frac{\sin \omega t - \omega \cos \omega t}{2\omega} )</td>
<td>( \frac{(e^{s\omega} - 1)}{s(e^{s\omega} + 1)} )</td>
<td>un reduced square</td>
</tr>
<tr>
<td>( tf(t) )</td>
<td>( F'(s) )</td>
<td>first derivative formula</td>
</tr>
<tr>
<td>( t^n f(t) )</td>
<td>( F^{(n)}(s) )</td>
<td>derivative formula</td>
</tr>
<tr>
<td>( \int_{0}^{t} f(\tau)d\tau )</td>
<td>( \frac{1}{s} \mathcal{L}{f(t)} )</td>
<td>transform of integral</td>
</tr>
<tr>
<td>( f(t) )</td>
<td>( \mathcal{L}{F(s)} )</td>
<td>derivative formula</td>
</tr>
<tr>
<td>( \frac{f(t)}{t} )</td>
<td>( \int_{0}^{\infty} \mathcal{L}{F(\sigma)}d\sigma )</td>
<td>integral formula</td>
</tr>
<tr>
<td>( f * g = \int_{0}^{t} f(\tau)g(t-\tau)d\tau )</td>
<td>( \mathcal{L}{f} \times \mathcal{L}{g} )</td>
<td>convolution</td>
</tr>
<tr>
<td>( f(t) = f(t+p) )</td>
<td>( \frac{1}{1-e^{-as}} \int_{0}^{p} e^{-st} f(t)dt )</td>
<td>periodic functions</td>
</tr>
<tr>
<td>( \delta_a(t) )</td>
<td>( \frac{1}{e^{-as}} \int_{0}^{a} e^{-st} f(t)dt )</td>
<td>delta function</td>
</tr>
</tbody>
</table>

Table 1: A Short List of Laplace Transforms

77. The Laplace Transform is a linear operator:

\[
\mathcal{L}\{af + \beta g\} = a\mathcal{L}\{f\} + \beta\mathcal{L}\{g\}.
\]

(71)

78. A function \( f \) is of exponential order as \( t \to \infty \) if there exist nonnegative constants \( M, c, \) and \( T \) such that \( |f(t)| \leq Me^{ct} \) for all \( t \geq T \).

79. A function is piecewise continuous if it is continuous except possibly at finitely many step discontinuities in any finite interval.
80. The Laplace Transform of a function $f$ exists if it is piecewise continuous and of exponential order. In that case we get
\[ \lim_{s \to \infty} F(s) = 0. \]  
(72)

81. The Laplace Transform is invertible. Note that the property (72) severely restricts the class of functions for which the inverse Laplace Transform is defined.

82. For our purposes, Laplace Transforms are useful because they can be used to convert differential equations into algebraic equations. Thus we convert the ODE, solve the algebraic problem, obtain the Laplace Transform of the solution of the ODE, and obtain that solution by applying the inverse Laplace Transform.

83. Unlike in the other solution techniques we have discussed, any initial conditions are built directly into the solution procedure.

84. A rational expression is **proper** if the degree of its numerator is less than that of its denominator. The phrase **Partial Fractions** refers to splitting a proper rational expression into a sum of simpler proper rational expressions. We write the split with as yet unknown coefficients in the numerator, recombine the expressions, and match terms in the numerators. There are shortcuts such as setting $s$ to specific values, or differentiating the resulting equations, but matching coefficients always works.

84.1 **Linear Partial Fractions:** Suppose $N(s)$ is a polynomial of degree less than $n$.

\[ \frac{N(s)}{(s - a)^n} = \frac{A_1}{s - a} + \frac{A_2}{(s - a)^2} + \ldots + \frac{A_n}{(s - a)^n} \]  
(73)

84.2 **Partial Fractions for irreducible square factors:** Suppose $N(s)$ is a polynomial of degree less than $2n$.

\[ \frac{N(s)}{((s - a)^2 + b^2)^n} = \frac{A_1 s + B_1}{(s - a)^2 + b^2} + \frac{A_2 s + B_2}{((s - a)^2 + b^2)^2} + \ldots + \frac{A_n s + B_n}{((s - a)^2 + b^2)^n} \]  
(74)

85. The **convolution** of two functions is defined by

\[ (f * g)(t) = f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)d\tau. \]  
(75)

86. The “**delta function**” $\delta_a$ is defined by the equation

\[ \int_0^{\infty} g(t)\delta_a(t)dt = g(a). \]  
(76)

It’s not a function in the strict sense, it is just a symbol with the property (76). We can also think of it as the derivative of the unit step function $u_a(t) = u(t - a)$ or as the limit of bell shaped functions with area 1 where the support shrinks to zero and the maximum function value goes to infinity.
\[ L \{ t e^{\frac{t^2}{2}} \} = e^{\frac{-s^2}{2}} - e^{\frac{(s-1)^2}{2}} \]

\[ = \int_0^\infty e^{-st} t e^{t} \, dt \]

\[ = \int_0^\infty \frac{e^{(1-s)t}}{(1-s)^2} \, dt \]

\[ = -\frac{1}{(1-s)^2} \int_0^\infty e^{(1-s)t} \, dt \]

\[ = -\frac{1}{(1-s)^2} \frac{1}{1-s} \]

\[ = \frac{1}{(1-s)^2} \]

\[ \mathbf{u} \cdot \mathbf{v} = \int_0^t u(\tau) v(t-\tau) \, d\tau \]

\[ L \{ u \ast v \} = \mathbf{u} \cdot \mathbf{v} \]

\[ u = t, \quad v = e^t \]

\[ \int_0^t e^{t-\tau} \, d\tau = \int_0^t e^{t-\tau} \, d\tau \]
\[ I_n = \int_0^t \tau^n e^{-\tau} \, d\tau = -\tau^n e^{-\tau}\bigg|_0^t + n \int_0^t \tau^{n-1} e^{-\tau} \, d\tau \]

\[ = -t^n e^{-t} + n I_{n-1} \quad \text{(*)} \]

\[ I_0 = \int_0^t e^{-\tau} \, d\tau = -e^{-\tau}\bigg|_0^t = -e^{-t} + 1 \]

\[ I_1 = -te^{-t} + 1(-e^{-t} + 1) \]

\[ = -e^{-t}(t+1) + 1 \]

\[ I_2 = -t^2 e^{-t} + 2\left(-e^{-t}(t+1) + 1\right) \]

\[ = -t^2 e^{-t} - 2e^{-t}(t+1) + 2 \]

\[ \int ((x+h) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x)}{i!} \times i \]
Logistic Equation

\[ x' = k \cdot (M - x) \]
\[ x(0) = x_0 \]

\[ x' = c \cdot x \quad \Rightarrow \quad x(t) = x_0 e^{ct} \]

\[ \frac{dx}{dt} = k \cdot (M - x) \quad \Rightarrow \quad x(0) = x_0 \]

\[ \frac{dx}{x(M-x)} = k \cdot dt \]

\[ \frac{1}{x(M-x)} = \frac{A}{x} + \frac{B}{M-x} \]

\[ = \frac{A(M-x) + Bx}{x(M-x)} \]

\[ A(M-x) + Bx = 1 \]
\[ x = 0 \quad A = \frac{1}{M} \]

\[ x = M \quad B = \frac{1}{M} \]

\[
\int \frac{dx}{x(M-x)} = \frac{1}{M} \int \frac{1}{x} + \frac{1}{M-x} \, dx
\]

\[
= \frac{1}{M} \left( \ln x - \ln(M-x) \right)
\]

\[
= \frac{1}{M} \ln \frac{x}{M-x}
\]

\[
\frac{1}{M} \ln \frac{x}{M-x} = kt + C_1
\]

\[
\ln \frac{x}{M-x} = Mkt + C_1
\]

\[
C_1 = \ln \frac{x_0}{M-x_0}
\]

\[
\frac{x}{M-x} = \frac{x_0}{M-x_0} e^{Mkt}
\]
\[ x = (M-x) \frac{x_0}{m-x_0} e^{Mt} \]

\[ x(1 + \frac{x_0}{m-x_0} e^{Mt}) = M \frac{x_0}{m-x_0} e^{Mt} \]

\[ x(t) = \frac{M x_0}{1 + \frac{x_0}{m-x_0} e^{Mt}} \]

\[ = \frac{M x_0}{(M-x_0) e^{-Mt} + x_0} \]
\[ y' = y - x \quad y(4) = 0 \quad y(-4) = \alpha \]
\[ y' = 1 \quad y = x + 1 \]

\[ y' - y = -x \quad e^{-x} \]
\[ e^{-x}(y' - y) = -xe^{-x} \]
\[ e^{-x}y = -xe^{-x} - \int e^{-x} \, dx \]
\[ = xe^{-x} + e^{-x} + c \]
\[ = e^{-x}(x + 1) + c \]
\[ y = x + 1 + ce^{x} \]

\[ x = 4 \quad y = 0 = 5 + c e^{4} \]
\[ y = x + 1 - \frac{5}{e^4} e^x \]

\[ y(-4) = -3 - \frac{5}{e^8} \approx -3 \]

This is consistent with the dfield plot on the next page.
\[ y' = y - x \]