Exam 2 will take place on Thursday, July 28. It will cover Chapters 5 and 6. You answer the questions on the exam itself, and you won’t be able to use notes, books, or any electronic devices. If a problem requires an arcane formula I will put it on the exam. If the answer is an arithmetic expression you don’t need to convert it to decimal form. (For example, write $\frac{\sqrt{2}}{3}$ instead of 0.471.)

You should have plenty of time to do the problems and should not be rushed. If you find yourself contemplating a lengthy calculation you may have missed an obvious simple step. To avoid disruption and distraction I won’t be able to answer questions during the exam. If you think there is something wrong with a problem write a note and if you are correct you will receive generous credit.

The exam questions will be taken verbatim (or perhaps simplified) from the exercises in chapters 5 and 6 in the textbook! Thus you want to go through those exercises, choose some to do before the exam, and make sure you can do any of them. Of course, because of time limits none of the exam questions will be very complicated.

You should be able to explain reasoning and concepts, as well as solve specific problems. In order to receive full or partial credit on any problem you must show all of your work and justify your conclusions.

The following list of specific topics in this review is neither complete nor self contained. Instead the items listed here should trigger your memory of many connected facts and concepts.

1. A system of differential equations is simply a set of several differential equations. In this course we only consider ordinary DEs, so there is only one independent variable, but there may be several dependent variables.

2. Any system of DEs can be converted to a first order system.

3. A system is autonomous if the derivatives do not depend explicitly on the independent variable.

4. A first order system can be converted to a high order system by elimination but the opposite process of converting a high order equation to a first order system is more common.

5. We considered linear systems of the form

$$x' = Ax + f(t)$$  \hspace{1cm} (1)

where $x$ and $f$ are vectors in $\mathbb{R}^n$ and $A$ is an $n \times n$ matrix. Its entries are usually constant, although the textbook does consider functions of $t$ in some places.

6. The system (1) is homogeneous if $f = 0$ and inhomogeneous otherwise. Thus the homogeneous problem is

$$x' = Ax.$$  \hspace{1cm} (2)

7. If $Av = \lambda v$, i.e., $v \neq 0$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$ then

$$x(t) = e^{\lambda t}v$$  \hspace{1cm} (3)

is a solution of (2).
8. If $A$ has $n$ linearly independent eigenvectors $v_i$ with corresponding real eigenvalues $\lambda_i$ then
the general solution of the homogeneous problem can be written as

$$x(t) = \sum_{\alpha_i} e^{\lambda_i t} v_i.$$ (4)

9. If $\lambda = p + qi$ is a complex eigenvalue with corresponding eigenvector $v = a + bi$ (where
$i^2 = -1$) then two linearly independent real solutions of the homogeneous problem (2) are

$$x_1(t) = e^{pt} (a \cos qt - b \sin qt)$$
and $$x_1(t) = e^{pt} (b \cos qt + a \sin qt)$$ (5)

10. An eigenvalue $\lambda$ of a matrix $A$ has **algebraic multiplicity** $a$ if it is a root of multiplicity $a$ of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$ (6)

$\lambda$ has **geometric multiplicity** $g$ if the dimension of the linear space spanned by the
eigenvectors corresponding to $\lambda$ is $g$. $\lambda$ is **complete** if $a = g$, and **defective** otherwise. The matrix $A$ is **defective** if the dimension of the space spanned by its eigenvectors is
less than $n$, and **non-defective** otherwise. A defective matrix has at least one defective
eigenvector.

11. The following table lists examples of various kinds of matrices:

<table>
<thead>
<tr>
<th></th>
<th>singular</th>
<th>non-singular</th>
</tr>
</thead>
</table>
| defective | \[
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}, \begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\] |

12. We did not discuss this in class, but as a quick reminder: symmetric matrices have real
eigenvalues, and their eigenvectors can be chosen so as to be orthogonal.

13. For defective eigenvalues we introduced the notion of **generalized eigenvectors**. Suppose
$\lambda$ is an eigenvalue with algebraic multiplicity $k$ and geometric multiplicity 1. Suppose also
that $v_1$ is an eigenvector corresponding to $\lambda$. Then $v_1, v_2, \ldots, v_k$ defined by

$$Av_1 = \lambda v_1$$
$$(A - \lambda I)v_2 = v_1$$
$$(A - \lambda I)v_3 = v_2$$
$$\vdots$$
$$(A - \lambda I)v_k = v_{k-1}$$ (8)

are generalized eigenvectors of $A$. 


14. If $\lambda$ is an eigenvalue of $A$ with geometric multiplicity $k$, and $v_1, v_2, \ldots, v_k$ is a chain of associated generalized eigenvectors then the following functions are $k$ linearly independent corresponding solutions:

\[
\begin{align*}
    x_1 &= e^{\lambda t}v_1 \\
    x_2 &= e^{\lambda t}(v_1 t + v_2) \\
    x_3 &= e^{\lambda t}\left(\frac{1}{2}v_1 t^2 + v_2 t + v_3\right) \\
    &\vdots \\
    x_k &= e^{\lambda t}\left(\frac{1}{(k-1)!}v_1 t^{k-1} + \ldots + v_{k-1} t + v_k\right)
\end{align*}
\]  

(9)

Notice the pattern: each factor multiplying the exponential is obtained by integrating the preceding factor, and using the new generalized eigenvector as the integration constant.

15. A **fundamental matrix** $\Phi(t)$ of the homogeneous system (2) is a non-singular matrix whose columns satisfy the differential equation (2). Any fundamental matrix also satisfies the differential equation

\[
\Phi'(t) = A\Phi(t).
\]  

(10)

16. The solution of the (constant coefficient linear homogeneous) initial value problem

\[
x' = Ax, \quad x(0) = x_0
\]  

(11)

can be written in terms of a fundamental matrix as

\[
x(t) = \Phi(t)\Phi(0)^{-1}x_0.
\]  

(12)

17. The **Matrix Exponential** (and many other matrix functions) can be defined in terms of power series. Specifically,

\[
e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.
\]  

(13)

18. In terms of a fundamental matrix $\Phi$ the matrix exponential can be computed as

\[
e^{At} = \Phi(t)\Phi(0)^{-1}.
\]  

(14)

19. In terms of the matrix exponential the unique solution of the initial value problem (11) can be written as

\[
x(t) = e^{At}x_0.
\]  

(15)

20. As for any linear problem, the solutions of the homogeneous problem (2) form a linear space (of dimension $n$) and the general solution of the problem (1) can be written as any particular solution at all plus the general solution of the homogeneous problem:

\[
x(t) = x_c(t) + x_p(t).
\]  

(16)
21. One method to find a particular solution is the **method of undetermined coefficients**. It works the same way as in the linear constant coefficient high order single equation case: write the particular solution as a linear combination of suitable terms, substitute in the original equation, match terms, and compute the coefficients.

22. The method of **Variation of Parameters** gives rise to the formula

\[ x_p(t) = \Phi(t) \int \Phi(t)^{-1} f(t) dt \]  

(17)

where \( \Phi(t) \) is a fundamental matrix for the homogeneous problem (2). The general solution of the inhomogeneous problem (1) can then be written as

\[ x(t) = \Phi(t)c + \Phi(t) \int \Phi(t)^{-1} f(t) dt. \]  

(18)

23. The solution of the initial value problem (11) can be written variously as

\[ x(t) = \Phi(t)\Phi(0)^{-1}x_0 + \Phi(t) \int_0^t \Phi(s)^{-1} f(s) ds \]

\[ = e^{At}x_0 + e^{At} \int_0^t e^{-As}f(s) ds \]  

(19)

24. Chapter 6 illustrates **nonlinear systems**, particularly the autonomous system of two equations

\[
\begin{align*}
x' &= F(x, y) \\
y' &= G(x, y)
\end{align*}
\]  

(20)

The solutions of this equation can be thought of as parameterizing curves in the \( x-y \) plane, which in this context is referred to as the **phase plane**. Those curves are **trajectories** (or **solution curves**) of the system. If the trajectories are closed they are also called **orbits**.

25. The fundamental idea of analyzing nonlinear systems is to **approximate them locally by linear systems**.

26. An **equilibrium point** of the nonlinear system (20) is a point \((x_0, y_0)\) for which

\[ x' = y' = 0. \]  

(21)

27. Suppose

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]  

(22)

is non-singular. Then the unique equilibrium point of the linear system

\[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]  

(23)
is the origin.

28. The behavior of the solutions of the system (23) depend on the eigenvalues of $A$. We consider the case that $t$ goes to infinity. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $A$, and $v_1$ and $v_2$ the corresponding eigenvectors.

29. Consider first the case that the eigenvalues are real. The following subcases are possible.

29.1 $\lambda_1 < \lambda_2 < 0$. The origin is an asymptotically stable improper nodal point. Incoming trajectories are parallel to $v_2$ at the origin.

29.2 $\lambda_1 > \lambda_2 > 0$. The origin is an unstable improper nodal point. Outgoing trajectories are parallel to $v_2$ at the origin.

29.3 $\lambda_2 > 0 > \lambda_1$. The origin is an unstable saddle point.

29.4 $\lambda_1 = \lambda_2 < 0$, $v_1$ and $v_2$ are linearly independent. The origin is a stable proper nodal point, all incoming trajectories are rays.

29.5 $\lambda_1 = \lambda_2 < 0$, $A$ is defective. The origin is a stable improper nodal point, all incoming trajectories are parallel to $v_1$.

29.6 $\lambda_1 = \lambda_2 > 0$, $v_1$ and $v_2$ are linearly independent. The origin is an unstable proper nodal point, all outgoing trajectories are rays.

29.7 $\lambda_1 = \lambda_2 > 0$, $A$ is defective. The origin is an unstable improper nodal point, all outgoing trajectories are parallel to $v_1$.

30. Suppose $\lambda_1$ and $\lambda_2$ are conjugate complex and $r$ is the real part of both. The following subcases are possible:

30.1 $r > 0$. The origin is an unstable spiral point.

30.2 $r < 0$. The origin is a stable spiral point.

30.3 $r = 0$. The origin is a stable equilibrium point. The trajectories are ellipses centered at the origin.

31. The linearization of the nonlinear problem (20) about the equilibrium point $(x_0, y_0)$ is the constant coefficient homogeneous linear system

$$
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  F_x(x_0, y_0) & F_y(x_0, y_0) \\
  G_x(x_0, y_0) & G_y(x_0, y_0)
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}.
$$

(24)

32. Sufficiently close to the equilibrium point the solutions of the nonlinear problem show the same qualitative behavior as the solution of its linearization.

33. The significance of assuming that the matrix in the linearization is non-singular is that the equilibrium point is isolated, i.e., there is a neighborhood of it that contains no other equilibrium points.

34. We considered various applications.

35. Mixture Problems. We have several interacting tanks. The concentration of each is one of the dependent variables.

36. Predator/Prey:

$$
\begin{align*}
  x' &= ax - pxy = x(a - py) & a, p > 0 \\
  y' &= -by + qxy = y(qx - b) & b, q > 0
\end{align*}
$$

(25)

There are two equilibrium points, the origin and $S = (b/q, a/p)$. The origin is an unstable saddle point, and $S$ is stable. The trajectories in the first quadrant form closed orbits around $S$. 

5
37. Competing Species

\[
x' = a_1 x - b_1 x^2 - c_1 x y = x(a_1 - b_1 x - c_1 y), \quad a_1, b_1, c_1 > 0 \\
y' = a_2 y - b_2 y^2 - c_2 x y = y(a_2 - b_2 y - c_2 x), \quad a_2, b_2, c_2 > 0
\] (26)

These were obtained from the logistic equation for each species by adding the competition terms \(c_1 xy\) and \(c_2 xy\). There are four equilibrium points corresponding to

37.1 The origin, no life.
37.2 \((0, a_2/b_2), (a_1/b_1, 0)\), only one species survives.
37.3 \(E = (x_E, y_E)\) where

\[
x_E = \frac{b_2 a_1 - c_1 a_2}{b_1 b_2 - c_1 c_2} \quad \text{and} \quad y_E = \frac{b_1 a_2 - c_2 a_1}{b_1 b_2 - c_1 c_2}
\] (27)

The two species coexist.

The coexistence point \(E\) is stable if \(c_1 c_2 < b_1 b_2\) and unstable if \(c_1 c_2 > b_1 b_2\).

38. Undamped Nonlinear Springs. We converted the nonlinear spring equation

\[
mx'' = -kx + \beta x^3
\] (28)

to the system

\[
x' = y \\
y' = -kx + \beta x^3
\] (29)

and visualized solutions in the Position-Velocity Phase Plane, the \(x - y\) plane. The spring is soft if \(\beta > 0\) and hard if \(\beta < 0\).

39. For the hard spring the origin is the only equilibrium point, its trajectories are closed ovals.
40. For the soft spring there are two additional equilibrium points. Both are unstable saddle points.
41. Damped Nonlinear Springs. The addition of a damping term led to

\[
mx'' = -cx' - kx + \beta x^3.
\] (30)

The origin becomes a nodal or spiral sink.

42. Damped Nonlinear Pendulum Oscillations led to the system

\[
x' = y \\
y' = -\omega^2 \sin x - cy
\] (31)

which in turn led to some pretty phase plane portraits.

43. We discussed chaos which is a rich subject recommended for further study.