Math 2280  
Summer 2016  
Exam 2 Summary

Exam 2 will take place on Thursday, June 30. It will cover Chapters 3 and 4. You answer the questions on the exam itself, and you won’t be able to use notes, books, or any electronic devices. If a problem requires an arcane formula I will put it on the exam. If the answer is an arithmetic expression you don’t need to convert it to decimal form. (For example, write $\sqrt{2}$ instead of 0.471.)

You should have plenty of time to do the problems and should not be rushed. If you find yourself contemplating a lengthy calculation you may have missed an obvious simple step.

To avoid disruption and distraction I won’t be able to answer questions during the exam. If you think there is something wrong with a problem write a note and if you are correct you will receive generous credit. 

The exam questions will be taken verbatim (or perhaps simplified) from the exercises in chapters 3 and 4 in the textbook! Thus you want to go through those exercises, choose some to do before the exam, and make sure you can do any of them. Of course, because of time limits none of the exam questions will be very complicated.

You should be able to explain reasoning and concepts, as well as solve specific problems. In order to receive full or partial credit on any problem you must show all of your work and justify your conclusions.

The following list of specific topics in this review is neither complete nor self contained. Instead the items listed here should trigger your memory of many connected facts and concepts.

- An operator $L$ is linear if
  $$L(u + v) = Lu + Lv \quad \text{and} \quad L(ku) = kLu$$
  for all $u$ and $v$ in its domain, and real scalars $k$.

- The general solution of a linear problem
  $$Lx = f$$
  is of the form
  $$x = x_p + x_c$$
  where $x_p$ is any particular solution of (3) and $x_c$, the complementary solution, is the general solution of the homogeneous problem
  $$Lx = 0.$$ (4)
  Thus the problem (3) is homogeneous if $f = 0$.

- The concept of linear independence of functions $f_1, \ldots, f_n$ is identical to that of the linear independence of vectors: The set $\{f_1, f_2, \ldots, f_n\}$ is linearly independent if
  $$\sum_{k=1}^{n} \alpha_k f_k = 0 \implies \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$ (5)

- We saw that the (sufficiently often differentiable) functions $f_1, f_2, \ldots, f_n$ are linearly independent if the Wronskian
  $$W = \begin{vmatrix}
  f_1 & f_2 & f_3 & \ldots & f_n \\
  f'_1 & f'_2 & f'_3 & \ldots & f'_n \\
  f''_1 & f''_2 & f''_3 & \ldots & f''_n \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \ldots & f_n^{(n-1)}
  \end{vmatrix} \neq 0.$$ (6)
  In this notation, the vertical bars indicate a determinant.

- We considered in particular $n$-th order constant coefficient linear differential equations of the form
  $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x)$$ (7)
  where the coefficients $a_0, \ldots, a_n$ are real constants (and $a_n \neq 0$).
To find the complementary solution $y_c(x)$, i.e., the solution of
\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \]  \hspace{1cm} (8)
we solve the **characteristic equation**
\[ p(r) = 0 \]  \hspace{1cm} (9)
where $p$ is the **characteristic polynomial**
\[ p(r) = a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0. \]  \hspace{1cm} (10)

Once we know the roots of $p$ we can construct $n$ linearly independent solutions of (8) using the following observations:
- if $r$ is a real root of $p$ then $y(x) = e^{rx}$ is a solution of (8).
- if $r = a + bi$ is a complex root of $p$ then its conjugate is also a root and the two functions
  \[ y = e^{ax} \cos x \quad \text{and} \quad y = e^{ax} \sin x \]  \hspace{1cm} (11)
are two solutions of (8).
- If $r$ is a (real or complex) root of multiplicity $m > 1$, i.e.,
  \[ p(r) = p'(r) = \ldots = p^{(m-1)}(r) = 0, \quad p^{(m)}(r) \neq 0 \]  \hspace{1cm} (12)
then we can construct additional solutions of (8) by multiplying the solutions obtained by the preceding rules with $x, x^2, \ldots, x^{m-1}$.
- The general solution of (8) is a linear combination of the form
  \[ y_c(x) = \sum_{k=1}^{n} \alpha_k y_k(x) \]  \hspace{1cm} (13)
where the $y_k$ are $n$ linearly independent functions constructed by the points above.
- To find a particular solution of (7) we considered two cases and two major methods:
- The **Method of Undetermined Coefficients**. It is applicable when the right hand side $f$ and *all of its derivatives* can be written as a linear combination of finitely many linearly independent functions. Examples for such functions $f$ include polynomials, exponentials, and linear combinations of sin and cos functions. Suppose those basis functions do not solve the homogeneous problem (8). Then let $y_p$ be a linear combination of those basis functions with undetermined but constant coefficients, substitute in (7), collect terms, and match them with the expansion of $f$ as a linear combination of those basis functions. You obtain a linear system of equations for the coefficients. If some of the basis functions do satisfy the homogeneous problem then multiply them with the least power of $x$ that makes them no longer satisfy the homogeneous equation.
- In the method of **variation of parameters** write the particular solution as a linear combination
  \[ y_p(x) = \sum_{k=1}^{n} u_k(x) y_k(x) \]  \hspace{1cm} (14)
where the linearly independent functions $y_k$ satisfy the homogeneous problem (8) and the coefficients $u_k(x)$ are functions of $x$ rather than constants. The coefficients satisfy the linear system
\[
\begin{bmatrix}
y_1 & y_2 & y_3 & \ldots & y_n \\
y_1' & y_2' & y_3' & \ldots & y_n' \\
y_1'' & y_2'' & y_3'' & \ldots & y_n'' \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \ldots & y_n^{(n-1)}
\end{bmatrix}
\begin{bmatrix}
u_1' \\
u_2' \\
u_3' \\
\vdots \\
u_n'
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\begin{bmatrix} f(x) \end{bmatrix}
\]  \hspace{1cm} (15)
Solve it and find the coefficients by integration.

- We learned how to convert a linear combination of \( \sin \) and \( \cos \) functions into a single \( \cos \) function:

\[
A \cos \omega t + B \sin \omega t = C \cos(\omega t - \alpha)
\]  

where

\[
C = \sqrt{A^2 + B^2} \quad \text{is the amplitude,}
\]
\[
\omega \quad \text{is the circular frequency,} \quad \frac{T}{2\pi} \quad \text{is the frequency, and} \quad \nu = \frac{1}{2\pi} \omega \quad \text{is the period},
\]
\[
\alpha = \begin{cases} 
\tan^{-1}(B/A) & \text{if } A > 0 \text{ and } B > 0 \\
\pi + \tan^{-1}(B/A) & \text{if } A < 0 \\
2\pi + \tan^{-1}(B/A) & \text{if } A > 0 \text{ and } B < 0
\end{cases} \quad \text{is the phase angle}
\]  

- A simple model of mechanical vibrations is the second order constant coefficient linear differential equation

\[
mx'' + cx' + kx = F(t).
\]

Here, \( x(t) \) is the displacement from equilibrium at time \( t \), \( m \) is the mass of the vibrating object, \( c \) is the damping constant, \( k \) is the spring constant, and \( F(t) \) is the external force acting on the system.

We saw several applications, and we know how to solve this differential equation.

- In the homogeneous problem,

\[
mx'' + cx' + kx = 0
\]

there are three possible cases, corresponding to the number of real solutions of the characteristic equation:

- **Overdamped**, \( c^2 > 4km \), 2 real solutions of \( p(r) = 0 \), \( x(t) \) is a linear combination of two decaying exponentials, \( x \) crosses the \( t \)-axis at most once.

- **Critically Damped**, \( c^2 = 4km \), 1 real solution of \( p(r) = 0 \), \( x(t) \) is a linear combination of \( e^{rt} \) and \( te^{rt} \) where \( r < 0 \). \( x \) crosses the \( t \)-axis at most once, and the graphs of \( x(t) \) look much like the graphs of the overdamped solutions.

- **Underdamped**, \( c^2 < 4km \), 2 conjugate complex solutions of \( p(r) = 0 \), \( x(t) \) is an oscillation multiplied with a decaying exponential. The graph of \( x(t) \) crosses the \( t \)-axis infinitely often.

- The simple pendulum gives rise to the equation

\[
\theta'' + \frac{g}{L} \sin \theta = 0
\]

which is of the form (18). The solution of the DE is an undamped harmonic motion. Our derivation illustrated two important techniques. The first was to use the principle of conservation of energy. The other, more minor, but frequently used, idea was to approximate the sine of an an angle by the angle itself:

\[
\sin \theta \approx \theta
\]

which is reasonable for small angles \( \theta \).

- In the inhomogeneous case we restricted our attention to simple harmonic external forces:

\[
mx'' + cx' + kx = F_0 \cos(\omega t).
\]

We discussed the phenomenon of resonance. This occurs when \( c = 0 \) and \( \omega \) equals the natural circular frequency of the system. In that case the amplitude grows linearly and without bound. In the presence of damping, \( c > 0 \), there is a circular frequency less than the natural frequency, for which the amplitude of the solution is maximized. This is called practical resonance.

- We briefly analyzed electric circuits which give rise to very similar differential equations as those discussed for mechanical vibrations.

- A major class of problems are boundary value problems where a specific solution of a differential equation is defined by conditions at more than one value of the independent variable. A typical problem is

\[
x'' = f(t, x, x'), \quad y(a) = A \quad \text{and} \quad y(b) = B.
\]
• Boundary value problems are very different from initial value problems. An initial value problem usually has a unique solution under reasonable smoothness assumptions. A boundary value problem may have none, one, or infinitely many solutions. Here are some examples:

\[
x'' = -x, \quad \begin{cases} x(0) = A, x(1) = B & \text{one unique solution} \\
 x(0) = 0, x(\pi) = 1 & \text{no solution}
\end{cases}
\]

(24)\[
x(0) = 0, x(\pi) = 0 & \text{infinitely many solutions}
\]

• An interesting and important special case of infinitely many solutions are eigenvalue problems. Consider a linear differential operator \( \Lambda \) with (suitably many) homogeneous boundary conditions. \( x \) is an eigenfunction of this problem with corresponding eigenvalue \( \lambda \) if \( x \neq 0 \) and

\[
\Lambda x = \lambda x \quad \text{and} \quad x \text{ satisfies the BCs.}
\]

(25)\[
\text{For example, suppose that} \quad \Lambda x = -x'', \quad x(0) = x(L) = 0.
\]

(26)\[
\text{The eigenfunctions and corresponding eigenvalues of this problem are}
\]

\[
x_k = \sin \frac{k\pi t}{L} \quad \text{and} \quad \lambda = - \left( \frac{k\pi}{L} \right)^2
\]

(27)\[
\text{where} \quad k = 1, 2, 3, 4, \ldots \quad \text{If} \ x \ \text{is an eigenfunction then any non-zero multiple of it is also an eigenfunction, with the same eigenvalue. The zero function is not an eigenfunction.}
\]

• Chapter 4 deals with Systems of Differential Equations. The phrase simply means that we have more than one equation, i.e., we have several dependent variables (but only one independent variable for an ODE).

• Any system of high order differential equations can be converted to a first order system of differential equations. The basic technique consists of just giving names to the derivatives, adding the definitions of those names to the system, and rewriting the original equations in terms of the new variables. For example, the second order equation

\[
x'' = f(t, x, x') \quad \text{is equivalent to the first order system} \quad \begin{cases} x' = y \\
y' = f(t, x, y) \end{cases}
\]

(28)\[
\text{and}
\]

• It is sometimes possible, and occasionally useful, to convert a system of differential equations to a single high order equation. We know in particular how to do this if the differential equations are constant coefficient linear. (The textbook refers to the associated differential operators as polynomial operators.) A crucial fact about those operators is that they commute. As a consequence we can use a variation of Cramer’s Rule. For example, consider the system

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{13} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\]

(29)\[
\text{The dependent variable} \quad x_1 \quad \text{satisfies the single high order equation}
\]

\[
\begin{vmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{13} \\
L_{31} & L_{32} & L_{33}
\end{vmatrix}
x_1 =
\begin{vmatrix}
f_1 & L_{12} & L_{13} \\
f_2 & L_{22} & L_{13} \\
f_3 & L_{32} & L_{33}
\end{vmatrix}
\]

(30)\[
\text{where the vertical bars indicate determinants. The other variables} \quad x_2 \quad \text{and} \quad x_3 \quad \text{satisfy analogous equations. The} \quad L_{ij} \quad \text{are polynomial operators of the form}
\]

\[
Lx = \sum_{k=0}^{n} a_k \frac{d^k}{dt^k} x
\]

(31)\[
\text{with constant coefficients} \quad a_k. \ \text{The conversion works more generally so long as the} \quad L_{ij} \quad \text{all commute.}
\[ y'' + y = xe^x \]

\[ y = y_p + y_c \]

\[ y_p = A e^x + B e^{-x} \quad A, B = ? \]

\[ y_p' = Ae^x + Be^{-x} \]

\[ y_p'' = Ae^x - Be^{-x} \]

\[ y_p'' = (2A + B)e^x + Ae^x \]

\[ y_p + y_p'' = 2A e^x + 2(A + B)e^x \]

\[ y_p + y_p'' = xe^x \]

\[ 2A = 1 \quad A = \frac{1}{2} \quad B = -\frac{1}{2} \]

\[ y_p = \frac{1}{2} (xe^x - e^x) \]

\[ = \frac{1}{2} e^x (x - 1) \]

\[ y_p' = \frac{1}{2} \left( e^x (x - 1) + e^x \right) = \frac{1}{2} x e^x \]

\[ y_p'' = \frac{1}{2} \left( e^x + xe^x \right) \]

\[ y_p + y_p'' = \frac{1}{2} (xe^x - e^x) + \frac{1}{2} (e^x + xe^x) = xe^x \]
\[ L y = F(x) \]
\[ F(x) = x^3 + 1 \]
\[ y_p = Ax^3 + Bx^2 + Cx + D \]

\[ L y = \cos x \]
\[ y_p = A \cos x + B \sin x \]
\[ x e^x \]

\[ x' = -x + 3 \]
\[ y' = 2y \]
\[ y = \frac{1}{3} (x' + x) \]
\[ y' = \frac{1}{3} (x'' + x') = 2y = \frac{2}{3} (x' + x) \]
\[ \frac{1}{3} x'' = \frac{1}{3} x' + \frac{2}{3} x \]
\[ x'' = x' + 2x \]
\[ x' = x - 2y \]
\[ y' = 2x - 3y \quad 2x = y' + 3y \]
\[ x = \frac{1}{2} y' + \frac{3}{2} y \]
\[ x' = \frac{1}{2} y'' + \frac{3}{2} y' = x - 2y = \frac{1}{2} y' + \frac{3}{2} y - 2y \]

\[ \frac{1}{2} y'' + y' + \frac{1}{2} y = 0 \]
\[ y'' + 2y' + y = 0 \]

\[ x' = \frac{3}{2} \]
\[ y' = \quad x = \]

\[ y'' + y = \cos x \quad y(0) = 1, \quad y'(0) = -1 \]

\[ y_p = x(A\cos x + B\sin x) \]
\[ y'_p = A\cos x + B\sin x \quad t(x(-A\sin x + B\cos x)) \]
\[ y''_p = -A\sin x + B\cos x - A\sin x + B\cos x \]
\[ = 2(-A\sin x + B\cos x) + x(-A\cos x - B\sin x) \]
\[ y_\rho'' + y_\rho = x (A \cos x + B \sin x) \]
\[ -x (4 \cos x + 8 \sin x) \]
\[ + 2 (-A \sin x + B \cos x) = \cos x \]
\[ B = \frac{1}{2}, \quad A = 0 \]

\[ y''' + 2y'' + 2y' + 3y = t \]

\[ u = y'' \]
\[ v = y' \]
\[ y''' = u' \]

\[ u' + 2u + 2v + 3y = t \]

\[ u' = t - 2u - 2v - 3y \]

\[ y'''' = f(t, y, y', y'', y''') \]

\[ u = y' \]
\[ y' = u \]
\[ v = y'' = u' \]
\[ u' = v \]
\[ w = y''' = v' \]
\[ v' = w \]

\[ w' = y'''' \]
\[ w' = f(t, y, u, v, w) \]
\[ \ddot{\theta} + \frac{g}{L} \theta = 0 \]

\[ \theta(t) = A \cos \omega t + B \sin \omega t \]

\[ \omega = \sqrt{\frac{g}{L}} \]

\[ x'' + 2x + y'' + 2y = e^{-3t} \]

\[ x'' + 3x \]

\[
\begin{vmatrix}
D+2 & D+2 \\
D+3 & D+3
\end{vmatrix}
\begin{vmatrix}
x \\
y
\end{vmatrix}
= 
\begin{vmatrix}
e^{-3t} \\
e^{-2t}
\end{vmatrix}
\begin{vmatrix}
D+2 \\
D+3
\end{vmatrix}
\]