This Week

- **Today**: Finish the discussion of new subject matter.
- **Tuesday**: I will attempt to summarize the whole semester in 60 minutes.
- **Wednesday**: Questions and Answers
- **Thursday**: no class
- **Friday**: 10:00-12:00, final exam, covering chapters 1:7. 10 questions.
• Pointer to chapter 8 (Power Series Methods).

• Express a function as a power series and do what you need to do to each term.

• Example:

\[
\mathcal{L}\{e^t\} = \frac{1}{s-1}.
\]

\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1
\]

\[
\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}
\]

\[
\mathcal{L}\{e^t\} = \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{t^n}{n!}\right\}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{L}\{t^n\}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{n!}{s^{n+1}}
\]

\[
= \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{1}{s}\right)^n
\]

\[
= \frac{1}{s} \cdot \frac{1}{1 - \frac{1}{s}} = \frac{1}{s} \cdot \frac{s}{s-1} = \frac{1}{s-1}
\]
Step Function and Shift of Origin

• We saw earlier that

\[ \mathcal{L}\{e^{at}f(t)\} = F(s-a) = \mathcal{L}\{f\}(s-a). \]

• Thus multiplying \( f \) with \( e^{at} \) causes a shift of origin by \( a \) on the \( s \)-axis in the Laplace Transform.

• This can also be interpreted as a shift on the \( t \)-axis in the original function.

• Let \( u \) be the step function defined by

\[ u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \]

and let

\[ u_a(t) = u(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases} \]

• Then we get, for \( a \geq 0 \):

\[ \mathcal{L}\{u(t-a)\} = \int_{0}^{\infty} e^{-st}u(t-a) = \int_{0}^{\infty} e^{-st} \, dt = \left. \frac{1}{s}e^{-as} \right|_{a}^{\infty} = \frac{e^{-as}}{s}. \]

• This is an example of a shift of origin on the \( t \)-axis.

• More generally:

**Theorem 1.** (page 474). If \( \mathcal{L}\{f(t)\} \) exists for for \( s > c \) then

\[ \mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \]

and

\[ \mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a). \quad s > a+c \]

• Note: We are not simply shifting \( f \), we are effectively requiring that \( f(t) \) be negative for negative \( f \).
\[ \mathcal{L} \{ u(t-a)f(t-a) \} = \int_{a}^{\infty} e^{-st} f(t-a) \, dt \]

\[ = e^{-sa} \int_{a}^{\omega} e^{-st} f(t-a) \, dt \]

\[ = e^{-sa} \int_{0}^{\infty} e^{-st} f(t) \, dt \]

\[ = e^{-sa} \mathcal{L} \{ f(t) \} \]
Example 3. Let

$$f(t) = \int_0^{2\pi} e^{-st} \cos 2tdt$$

Compute its Laplace Transform.

Approach 1. Using Integration by parts twice gives (exercise, we did this many times . . .)

$$\mathcal{L}\{f(t)\} = \int_0^{2\pi} e^{-st} \cos 2tdt = \frac{s(1 - e^{-2\pi s})}{s^2 + 4}$$

• Approach 2. Use that

$$f(t) = [1 - u(t - 2\pi)] \cos 2t.$$
Periodicity

- A function is periodic of periodicity \( p \) if
  \[
  f(t + p) = f(t)
  \]
  for all \( t \).

- For example, \( \sin \) and \( \cos \) are periodic of periodicity \( 2\pi \). They are also periodic of periodicity \( 4\pi \) or \( 10^{20}\pi \). The tan function is periodic of periodicity \( \pi \) but also of periodicity \( 2\pi \), for example.

- Periodic functions are often more complicated than \( \sin \) and \( \cos \).

- Pointer to Chapter 9 (which is otherwise beyond our scope): Write a periodic function as a series whose terms are \( \sin \) and \( \cos \) functions (i.e., a Fourier Series) and then do what you want to do to the function term by term.

- We computed several examples of the Laplace Transform. In all cases we obtained a geometric series. This can be done very generally.

**Theorem 2.** (p. 478) Let \( f(t) \) be periodic with period \( p \) and piecewise continuous for \( t \geq 0 \). Then the transform

\[
F(s) = \mathcal{L}\{f(t)\}
\]

exists for \( s > 0 \) and is given by

\[
F(s) = \frac{1}{1 - e^{-ps}} \int_{0}^{\infty} e^{-st} f(t) dt
\]

- Thus we only need to integrate over one period.
\[ \mathcal{L} \{ f(t) \} = \int_{0}^{\infty} e^{-st} f(t) \, dt = \sum_{n=0}^{\infty} \int_{0}^{\infty} e^{-s(t+np)} f(t) \, dt \\
= \sum_{n=0}^{\infty} \int_{0}^{\infty} (e^{-sp})^n f(t) \, dt \\
= \sum_{n=0}^{\infty} \left( \frac{1}{1 - e^{-ps}} \right)^n \int_{0}^{\infty} e^{-s(t+np)} f(t) \, dt \]

\[ = \frac{1}{1 - e^{-ps}} \int_{0}^{\infty} e^{-s(t+np)} f(t) \, dt \]
• Let’s check the formula on a known example.

\[ \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1} \]

\[ p = 2\pi \]

\[ \mathcal{L}\{\sin^2 t\} = \frac{1}{1 - e^{-2\pi s}} \int_0^\infty e^{-st} \sin t \, dt \]

\[
= \frac{1}{1 - e^{-2\pi s}} \left( -\frac{1}{5} e^{-st} \left( e^{2\pi s} - 1 \right) - \frac{1}{5} \int_0^\infty e^{-st} \cos t \, dt \right)
\]

\[
= \frac{1}{1 - e^{-2\pi s}} \cdot \frac{1}{5} \left( -\cos \left( \frac{1}{5} e^{2\pi s} \right) - \frac{1}{5} \int_0^\infty e^{-st} \cos t \, dt \right)
\]

\[ I = \int_0^\infty e^{-st} \sin t \, dt = \frac{1}{s^2} \left( e^{-2\pi s} - 1 \right) - \frac{1}{s^2} e^{-2\pi s} I + 1
\]

\[ (s^2 + 1) I = -e^{-2\pi s} + 1
\]

\[ I = \frac{-e^{-2\pi s} + 1}{s^2 + 1}
\]

\[ \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1} \]
Delta Functions

• “A delta function is a function that is zero everywhere except in one point where it is infinite so thoroughly that its integral equals 1.”

• Consider for example a golf club striking a golf ball. The club and ball are in contact for a very short period, and yet the ball flies away.

• To simplify things we think of the contact time as being zero (but that of course is non-physical).

• The definition of $\delta_a(t)$ (for $a \geq 0$, say) is that

$$\int_0^\infty g(t)\delta_a(t)dt = g(a).$$

• There is no such function, of course, but nonetheless the symbol $\delta_a(t)$ is called a delta function.

• Interpretation as the limit of functions.

$$\lim_{n \to \infty} \delta_n(t) = \delta_a(t)$$

• Interpretation as the derivative of a step function

• In any case:

$$\mathcal{L}\{\delta_a\} = \int_0^\infty e^{-st}\delta_a(t)dt = e^{-as}.$$

• Thus we can model a very short driving impulse in the Laplace Transform of a differential equation.

• Very simple example. Undamped Spring Mass system:

$$x'' + x = \delta_0(t), \quad x(0) = x'(0) = 0.$$
\[ \mathcal{L}\{x''\} = s^2 \overline{x} \quad \overline{x} = \mathcal{L}\{x'\} \]
\[ s^2 \overline{x} + \overline{x} = e^{-s \cdot 0} = 1 \]
\[ \overline{x} = \frac{1}{s^2 + 1} \]
\[ \mathcal{L}^{-1}\{\overline{x}\} = \sin t \]
\[ \mathcal{L}\{x''\} = s^2 \overline{x} - s \cdot \overline{x}(0) - x(0) \]
The End