6.4 Nonlinear Mechanical Systems

Recall our mass spring system illustrated in Figure 1 (taken from https://thiscondensedlife.wordpress.com/tag/harmonic-oscillator/).

Assuming Hooke’s Law

\[ F = -kx \]

the system gives rise to the DE

\[ mx'' = -kx \]

Let’s ask what happens if the force \( F \) is not linear.

We know that \( F(0) = 0 \) since \( x \) is the departure from equilibrium.

Let’s also suppose the Spring is symmetric, i.e.,

\[ F(x) = -F(-x). \]

In other words, \( F \) is odd.
Then we can write $F$ as a power series

$$F(x) = -kx + \beta x^3 + \beta_5 x^5 + \beta_7 x^7 + \ldots.$$ 

Finally, let’s ignore the higher order terms and assume

$$F(x) = -kx + \beta x^3.$$ 

Thus we have the second order DE

$$mx'' = -kx + \beta x^3.$$ 

So that we can use a phase plane portrait, let’s convert to a first order system by writing $y = x'$.

We get

$$x' = y$$

$$my' = -kx + \beta x^3$$

The phase plane portrait plots the location $x$ against the velocity $y$.

We can solve explicitly for trajectories of this system:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-kx + \beta x^3}{my}$$

This is a separable DE that can be solved easily for $y = y(x)$. (Exercise!)

The textbook proceeds differently.

Rewrite (1) as

$$mydy = -kx + \beta x^3dx.$$ 

Integrate
\[ \frac{1}{2} my^2 = -\frac{1}{2} kx^2 + \frac{\beta}{4} x^4 + E. \]

- rearrange

\[ \frac{1}{2} my^2 + \frac{1}{2} kx^2 - \frac{\beta}{4} x^4 = E. \]

The variable \( E \) stands for “energy”.

- It’s the sum of kinetic energy \( \frac{1}{2} my^2 \) and the potential energy \( \frac{1}{2} x^2 - \frac{1}{4} \beta x^4 \) stored in the expanded or compressed spring.

- Energy is conserved! When \( x = 0 \) it’s all kinetic,

\[ E = \frac{1}{2} my^2. \]

- The behavior of the system depends on the nature of the nonlinearity.

- The spring is soft if \( \beta > 0 \) and hard if \( \beta < 0 \).

\[ x' = y \]

\[ m y' = -kx + \beta x^3 = -x (k - \beta x^2) \]

CP: \( \beta < 0 \) \( (0,0) \)

\[ \beta > 0 \quad y=0 \]
Hard Spring, $\beta < 0$

\[ x = \pm \sqrt{\frac{k}{\beta}} \]

\[ (0, 0), \left( \frac{\sqrt{\frac{k}{\beta}}}{\beta}, 0 \right), \left( -\frac{\sqrt{\frac{k}{\beta}}}{\beta}, 0 \right) \]

\[ x' = y \]
\[ y' = -kx + \beta x^3 \]

\[
\begin{bmatrix}
0 & 1 \\
-k + 3\beta x^2 & 0
\end{bmatrix}
\]

\[ J(0, 0) = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \]

\[ \left| J(0, 0) - \lambda I \right| = \lambda^2 + k \]

\[ \lambda = \pm \sqrt{k \beta^2} \]

\[ x = \sqrt{\frac{k}{\beta}} \]

\[ -k + 3\beta x^2 = -k + \frac{3k\beta}{\beta^3} = 2k \]
\beta > 0 \quad (\pm \sqrt{\frac{k}{\beta}}, 0)

\gamma = \begin{bmatrix} 0 & 1 \\ 2k & 0 \end{bmatrix}

\left| \gamma - \gamma \right| = \lambda^2 - 2k

\lambda = \pm \sqrt{2k}

saddle point
Soft Spring, $\beta > 0$
Damping

- add a dashpot
- We get the second order equation

\[ mx'' = -cx' - kx + \beta x^3 \]

and the equivalent first order systems

\[ x' = y \]
\[ y' = -\frac{c}{m} y - \frac{k}{m} x + \frac{\beta}{m} x^3 \quad m = 1 \]

\[ \begin{bmatrix} O & 1 \\ -k + 3\beta x^2 & -c \end{bmatrix} \]

\[ \left| J - \lambda I \right| = -\lambda (-c - \lambda) + k - 3\beta x^2 \]
\[ = \lambda^2 + c\lambda + k - 3\beta x^2 \]
\[ \lambda = 0 \quad \lambda = \frac{\sqrt{k}}{B} \]
\[ x = 0 \quad k < 2k \]
\[ x = \sqrt{\frac{k}{B}} \]

\[ x = 0 \quad \lambda^2 + c\lambda + k = 0 \]
\[ \lambda = \frac{-c \pm \sqrt{c^2 - 4k}}{2} \]
A nonlinear Pendulum

- Consider a pendulum of length $L$ making an angle $\theta$ with the vertical. In section 3.4 we derived the nonlinear equation

$$\theta'' + \frac{g}{L} \sin \theta = 0$$

- We then analyzed the linearized system by replacing $\sin \theta$ with $\theta$.

- Today let's look at the nonlinear version and also allow for damping.

- $$\theta'' + c\theta' + \omega^2 \sin \theta = 0 \quad \text{where} \quad \omega^2 = \frac{g}{L}.$$ 

- Letting $x = \theta(t)$ and $y = \theta'(t)$ gives the system

$$\begin{align*}
x' &= y \\
y' &= -cy - \omega^2 \sin x
\end{align*}$$

\[ CP: \quad y = 0 \quad y' = -\omega^2 \sin x \]

\[ x = \pi \overline{n} \]

\[ J = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & -c \end{bmatrix} \]
\[ |J - \lambda I| = -\lambda(\lambda - \lambda) + \omega^2 \cos x \]
\[ = \lambda^2 + \omega^2 \cos x = 0 \]
\[ \lambda = \frac{-c \pm \sqrt{c^2 + 4 \omega^2}}{2} \]

\begin{align*}
  n & = \text{odd} \quad \frac{-c \pm \sqrt{c^2 + 4 \omega^2}}{2} \\
  n & = \text{even} \quad -c \pm \sqrt{c^2 - 4 \omega^2} \\
\end{align*}