6.4 Nonlinear Mechanical Systems

- Notes available on

  http://www.math.utah.edu/~pa/2280/34.pdf

  Will replace them with the annotated version after class.

![Mass Spring System](https://thiscondensedlife.wordpress.com/tag/harmonic-oscillator/)

**Figure 1.** Mass Spring System.

- Recall our mass spring system illustrated in Figure 1 (taken from https://thiscondensedlife.wordpress.com/tag/harmonic-oscillator/).

- Assuming Hooke’s Law

  \[ F = -kx \]

  the system gives rise to the DE

  \[ mx'' = -kx \]

- Let’s ask what happens if the force \( F \) is not linear.

- We know that \( F(0) = 0 \) since \( x \) is the departure from equilibrium.
• Let’s also suppose the Spring is symmetric, i.e.,

\[ F(x) = -F(-x). \]

• In other words, \( F \) is odd.

• Then we can write \( F \) as a power series

\[ F(x) = -kx + \beta x^3 + \beta_5 x^5 + \beta_7 x^7 + \ldots. \]

• Finally, let’s ignore the higher order terms and assume

\[ F(x) = -kx + \beta x^3. \]

• Thus we have the second order DE

\[ mx'' = -kx + \beta x^3. \]

• So that we can use a phase plane portrait, let’s convert to a first order system by writing \( y = x' \).

• We get

\[ x' = y \]
\[ my' = -kx + \beta x^3 \]

• The phase plane portrait plots the location \( x \) against the velocity \( y \).

• We can solve explicitly for trajectories of this system:

\[ \frac{m dy/dt}{m dx/dt} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-kx + \beta x^3}{my} \tag{1} \]

• This is a separable DE that can be solved easily for \( y = y(x) \).
• Rewrite (1) as

\[ mydy = -kx + \beta x^3 dx. \]

• Integrate

\[ \frac{1}{2} my^2 = -\frac{1}{2} kx^2 + \frac{\beta}{4} x^4 + E. \]

• rearrange

\[ \frac{1}{2} my^2 + \frac{1}{2} kx^2 - \frac{\beta}{4} x^4 = E. \] (2)

The integration constant \( E \) stands for “energy”.

• It’s the sum of kinetic energy \( \frac{1}{2} my^2 \) and the potential energy \( \frac{1}{2} x^2 - \frac{1}{4} \beta x^4 \) stored in the expanded or compressed spring.

• Energy is conserved! When \( x = 0 \) it’s all kinetic,

\[ E = \frac{1}{2} my^2. \]

• We can solve the energy equation for \( y \) (velocity) in terms of \( x \) (location).

\[ y = \pm \sqrt{\frac{2E}{m} - \frac{kx^2}{m} + \frac{\beta}{2m} x^4}. \]

• The behavior of the system depends on the nature of the nonlinearity, specifically on the sign of \( \beta \).

• The spring is soft if \( \beta > 0 \) and hard if \( \beta < 0 \).
• In particular, the number of critical points depends on the sign of $\beta$

$$\dot{x} = y$$
$$\dot{y} = -kx + \beta x^3$$
$$y = 0$$

$\beta < 0$:

$$\dot{y} = -kx - 1\beta x^3$$
$$= -x(k + 1\beta x^2) = 0 \Rightarrow x = 0$$

$(0,0)$ CP

$\beta > 0$ soft spring $y = 0$

$$\dot{x} = y$$
$$\dot{y} = -kx + 1\beta x^3$$
$$= x(1\beta x^2 - k) = 0 \Rightarrow x = 0$$

$$x = \pm \sqrt{\frac{k}{\beta}}$$

$(0,0)$ \((\pm \sqrt{\frac{k}{\beta}}, 0)\)

$$\mathbb{J}(x, y) = \begin{bmatrix} 0 & 1 \\ \frac{1}{m}(-4 + 3\beta x^2) & 0 \end{bmatrix}$$
The nature of the critical points can be determined by computing the eigenvalues of the Jacobian of the system

\[ x' = y \]
\[ y' = -\frac{k}{m}x + \frac{\beta}{m}x^3 \]

We get

\[ J(x, y) = \begin{bmatrix} 0 & \frac{3\beta}{m}x^2 & 1 \\ -\frac{k}{m} + \frac{3\beta}{m}x^2 & 0 \end{bmatrix} \]

In particular

\[ J(0, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \]

\[ |J - \lambda I| = \lambda^2 + \frac{k}{m} \]

\[ \lambda = \pm \sqrt{\frac{k}{m}} \]

and, for soft springs with \( \beta > 0 \),

\[ J \left( 0, \pm \sqrt{\frac{k}{\beta}} \right) = \begin{bmatrix} 0 & 1 \\ \frac{2k}{m} & 0 \end{bmatrix} \]

\[ |J - \lambda I| = \lambda^2 - \frac{2k}{m} \]

\[ \lambda = \pm \sqrt{\frac{2k}{m}} \]

saddle points
Figure 2. Hard nonlinear spring, $m = 1$, $k = 1$, $\beta = -1$. 
Figure 3. Soft nonlinear spring, $m = 1$, $k = 1$, $\beta = 1$.

**Damping**

- add a dashpot
- We get the second order equation

$$mx'' = -cx' - kx + \beta x^3$$

and the equivalent first order systems

$$x' = y$$

$$y' = -\frac{c}{m} y - \frac{k}{m} x + \frac{\beta}{m} x^3$$
and can repeat the analysis

\[ Y = \begin{bmatrix} 0 & 1 \\ \frac{k}{m} + \frac{3Bx^2}{m} & -\frac{c}{m} \end{bmatrix} \]

\[ |J-xI| = -2(-\lambda - \frac{c}{m}) - \left(\frac{k}{m} + \frac{3Bx^2}{m}\right) \]

\[ x=0: \quad \lambda^2 + \frac{c}{m} \lambda + \frac{k}{m} = 0 \]

\[ \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ \lambda = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - \frac{4k}{m}}}{2} \]

same CPs
Figure 4. Damped hard nonlinear spring, $m = 1, k = 1, \beta = -1$. 
Figure 5. Damped soft nonlinear spring, $m = 1$, $k = 1$, $\beta = 1$.

**A nonlinear Pendulum**

- Consider a pendulum of length $L$ making an angle $\theta$ with the vertical. In section 3.4 we derived the nonlinear equation
  
  $$\theta'' + \frac{g}{L} \sin \theta = 0$$

- We then analyzed the linearized system by replacing $\sin \theta$ with $\theta$.

- Today let’s look at the nonlinear version and also allow for damping.

- $\theta'' + c\theta' + \omega^2 \sin \theta = 0$ where $\omega^2 = \frac{g}{L}$.

- Letting $x = \theta(t)$ and $y = \theta'(t)$ gives the system
\[ x' = y \]
\[ y' = -cy - \omega^2 \sin x \]

- Again, we can analyze critical points and their nature.

\[ \text{CP: } y = 0 \]

\[ c = 0 \]

\[ \omega^2 \sin x = 0 \]

\[ x = n\pi \quad n \text{ integer} \]

\[ Y = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & -c \end{bmatrix} \quad c = 0 \]

\[ |J - \lambda I| = \lambda^2 + \omega^2 \cos x \]

\[ \lambda = \pm \omega \sqrt{\cos x} \]
Figure 6. Undamped nonlinear pendulum, $\omega = 1, c = 0$. 
Figure 7. Damped nonlinear pendulum, $\omega = 1, c = 0.2$. 

$x' = y$
$y' = -cy - f \sin(x)$

c = 0.2
f = 1