3.4 Mechanical Vibrations

- This is an application of what we discussed last week, specifically the solutions of constant coefficient linear second order DEs.
- We’ll spend an hour, but there are books on the subject!

![A mass-spring-dashpot system.](image)

**Figure 1.** A mass-spring-dashpot system.

- Consider the mass-spring-dashpot system in Figure 1.
- The mass moves back and forth (say, without gravity or friction) under the influence of the Spring and the Dashpot (like a shock absorber in your car). All forces are horizontal.
- Let \( x = x(t) \) be the deviation of the location of the mass from equilibrium.
- The spring acts on the mass with a force that is proportional and opposed to \( x \).
- The dashpot acts on the mass with a force that is proportional and opposed to \( x' \).
- According to Newton, force equals mass times acceleration.
- We get the DE
\[ mx'' = -kx - cx' + F(t) \]

where \( F(t) \) is an external force.

- This can be rewritten as a linear second order constant coefficient DE:
  \[ mx'' + kx + cx' = F(t). \]  \hspace{1cm} (1)

- Note that we can interpret the whole assembly as being vertical. Gravity would only change the natural length of the spring.

- There are other situations that give rise to this kind of DE. For example:
A simple Pendulum

- Let \( s \) be the length of the arc from the bottom to the current location of \( m \).
- Energy = potential energy \((mgh)\) + kinetic energy \(\left(\frac{mv^2}{2}\right)\).
- Energy is conserved!
- Thus we get the DE
\[
\frac{1}{2}ms^2 + mgh = E.
\]

- We want to express everything in terms of the angle \( \theta \) as a function of \( t \).
- Note that

\[
\frac{L - h}{L} = \cos \theta
\]

- Thus

\[
h = L(1 - \cos \theta)
\]

- Moreover

\[
s = \theta L \quad \text{and} \quad s' = \theta' L
\]

- Our DE becomes

\[
\frac{1}{2}m\theta'^2L^2 + mgL(1 - \cos \theta) = E.
\]

- This is a nonlinear first order DE that we could attempt to solve. But it’s inconvenient to have the energy as a constant, and also, more subtly, today we want a second order linear DE!

- Differentiating with respect to \( t \) gives

\[
m\theta'\theta''L^2 + mgL\theta' \sin \theta = 0.
\]

- Dividing by \( \theta'L^2m \) gives

\[
\theta'' + \frac{g}{L} \sin \theta = 0.
\]

- This equation is still not linear!
• But, for small values of $\theta$,

$$\theta \approx \sin \theta.$$  

• So our equation is approximately the same as

$$\theta'' + \frac{g}{L} \theta = 0$$  \hspace{1cm} (2)

• You may think it’s audacious to approximate the nonlinear DE with a linear one, but it’s commonplace to do so and happens all the time!

• The solution of (2) is obvious:

$$\Theta = A \cos \sqrt{\frac{g}{L}} t + B \sin \sqrt{\frac{g}{L}} t.$$  

• Note that the DE (2) and its solution are independent of $m$!
• The equation
\[ \theta'' + \frac{g}{L} \theta = 0 \]  

is a special case of the equation
\[ m \ddot{x} + kx + c \dot{x} = F(t). \]

which we got earlier.

• Let’s now study (4) in general.

**Simple Harmonic Motion**

• Suppose there is no damping. We get
\[ m \ddot{x} + kx = 0 \]

which has the solution
\[ x'' = -\frac{k}{m} x \]

\[ x = A \cos \sqrt{\frac{k}{m}} t + B \sin \sqrt{\frac{k}{m}} t \quad \omega_0 = \sqrt{\frac{k}{m}} \]

\[ = C_1 \cos (\omega_0 t - \alpha) \]

\[ = C_1 \left( \cos \omega_0 t \cos \alpha + \sin \omega_0 t \sin \alpha \right) \]

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- Use the trig identity

\[ \cos(x - y) = \cos x \cos y + \sin x \sin y. \]
\[ A = C \cos \theta \quad \Rightarrow \quad \sqrt{A^2 + B^2} \]
\[ B = C \sin \theta \]
\[ \tan \theta = \frac{B}{A} \]

\[ \alpha = \arctan \frac{B}{A} \]
\[ \alpha = 2\pi + \arctan \frac{B}{A} \]
• The parameters in

\[ x(t) = C \cos(\omega_0 t - \alpha) \]

all have names:

- \( C \) is the **amplitude**
- \( T = \frac{2\pi}{\omega_0} \) is the **period**, the time for one oscillation.
- \( \frac{\omega_0}{2\pi} \) is the **frequency**
- \( \omega_0 \) is the **circular frequency**.
- \( \alpha \) is the **phase angle**.
- \( \frac{\alpha}{\omega_0} \) is the **time lag**.

• Note that frequency is independent of the amplitude. That’s the principle underlying a grandfather clock. On the other hand, we obtained the DE by linearizing when we approximated \( \sin \theta \) with \( \theta \). That’s why grandfather clocks always have pendulums that swing with a small amplitude.
General Solution of $m x'' + c x' + k x = 0$

**Characteristic Eqn**

$$m \quad 2 + c r + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

If $c^2 - 4km > 0$,

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} < 0$$

$$r_2 = \frac{-c - \sqrt{c^2 - km}}{2m} < 0$$

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

**Overdamped**
\[ c^2 - km = 0 \]

\[ \tau = -\frac{c}{2m} < 0 \quad \text{double root} \]

\[ x(t) = A e^{\tau t} + B te^{\tau t} \]

\[ = (A + Bt)e^{\tau t} \quad \rightarrow 0 \text{ as } t \rightarrow \infty \]

SDH

\[ c^2 - km < 0 \quad \text{underdamped} \]

\[ \tau = \frac{-c + \sqrt{c^2 - km}}{2m} \]

\[ = p + \omega i \quad p = -\frac{c}{2m} \]

\[ \omega = \frac{\sqrt{4km - c^2}}{2m} \]
\[ x(t) = e^{-\frac{ct}{2m}} (A \cos \omega t + B \sin \omega t) \]
3.5 Nonhomogeneous Equations

The Method of Undetermined Coefficients

• Example 1:

\[ y'' + 3y' + 4y = 3x + 2 \quad y = y(x). \]

\[ r^2 + 3r + 4 = 0 \quad r = \frac{-3 \pm \sqrt{9 - 16}}{2} \]

\[ 0 + 3A + 4(Ax + B) = 3x + 2 \]

\[ 4Ax + (3A + 4B) = 3x + 2 \]

\[ 4A = 3 \quad \frac{9}{4} + 4B = 2 \]

\[ A = \frac{3}{4} \quad 4B = \frac{1}{4} \]

\[ 4B = \frac{1}{4} \quad B = -\frac{1}{16} \]

\[ y_p = \frac{3}{4}x - \frac{1}{16} \]
• Example 2:

\[ y'' - 4y = 2e^{3x}. \]

\[ \gamma_p(x) = Ae^{3x} \]

\[ \gamma''_p(x) = 9Ae^{3x} \]

\[ 9Ae^{3x} - 4Ae^{3x} = 2e^{3x} \]

\[ 5A = 2 \]

\[ A = \frac{2}{5} \]

\[ y(x) = \frac{2}{5}e^{3x} + Ae^{2x} + Be^{-2x} \]