General Solution of Linear Problems

- Very Major Principle: The general solution of a linear problem is any particular solution, plus the general solution of the corresponding homogeneous problem.

- Of course we saw yesterday how to compute the general solution of a linear DE, at least in the case that the coefficients are constant.

- The principle holds in great generality.

- For example, linear systems:

\[ Ax = b \]

\[ A \tilde{x}_p = b \]

\[ A \tilde{x}_c = 0 \]

\[ A(x_p + x_c) = A x_p + A x_c = b + 0 = b \]
**Theorem 5**, page 158 (abridged). Let $y_p$ be a particular solution of
\[ D y = y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = f(x). \] (1)

Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous problem
\[ y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0. \] (2)

Let $Y$ be any solution whatsoever of (1). Then there exists constants $c_1, c_2, \ldots, c_n$ such that
\[ Y = y_p + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n. \] (3)

- The textbook calls
\[ y_c = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n. \] (4)
the complementary function.

- With that notation we get
\[ y = y_p + y_c \]
• How do we recognize that functions are linearly independent?

• Use the Wronskian.

**Theorem 3**, page 156. Suppose \( y_1, y_2, \ldots, y_n \) are solutions of (2) on some interval \( I \). Let

\[
W(x) \leq W = \begin{vmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
\]

where the bars indicate a determinant. (\( W \) is the Wronskian of the set of solutions.) Then

(a) If \( y_1, y_2, \ldots, y_n \) are linearly dependent then \( W(x) = 0 \) for all \( x \) in \( I \).

(b) If \( y_1, y_2, \ldots, y_n \) are linearly independent then \( W(x) \neq 0 \) for all \( x \) in \( I \).

\[
W(a) = 0 \\
y(x) = \sum c_i y_i \\
y'(a) = 0 \\
y''(a) = 0 \\
\vdots \\
y^{(n-1)}(a) \neq 0
\]
• linear independence of $e^{rx}$ and $e^{sx}$.

$$W = \begin{vmatrix} e^{rx} & e^{sx} \\ re^{rx} & se^{sx} \end{vmatrix} = e^{rx}se^{sx} - e^{sx}re^{rx}$$

$$= e^{(r+s)x}(s-r)$$

• linear independence of $e^x$ and $xe^x$.

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix}$$

$$= e^x e^x(1+x) - e^x e^x x$$

$$= e^{2x} (1+x-x) = e^{2x} \neq 0$$
Example 5:

\[ x^3 y''' - x^2 y'' + 2xy' - 2y = 0 \]

has these solutions

\[
\begin{align*}
y_1(x) &= x \\
y_2(x) &= x \ln x \\
y_3(x) &= x^2
\end{align*}
\]

(Exercise: check by substitution in DE).

Verify that these solutions are linearly independent.

\[
W = \begin{vmatrix}
x & x \ln x & x^2 \\
1 & 1 + \ln x & 2x \\
0 & \frac{1}{x} & 2
\end{vmatrix}
\]

\[
= -\frac{1}{x} \left[ x^2 \right] + 2 \left[ x + x \ln x - x (\ln x) \right]
\]

\[
= -\frac{1}{x} \left( 2x^2 - x^2 \right) + 2 \left( x + x \ln x - x (\ln x) \right)
\]

\[
= -\frac{1}{x} x^2 + 2x = 2x - x = x
\]
General Solution of homogeneous constant coefficient \( n \)-th order linear DEs

A root \( r \) is a root of multiplicity \( p \) of a function \( f \) if

\[ f(r) = f'(r) = \ldots = f^{(p-1)}(r) = 0 \quad \text{and} \quad f^{(p)}(r) \neq 0. \]

- DE:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0. \]

- Characteristic equation

\[ p(r) \sim a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0. \]

- Suppose \( r_1, r_2, \ldots, r_n \) are the roots of the characteristic equation.

- Then linearly independent solutions of (2) can be constructed as follows:

  - \( x^i e^{rx}, i = 0, \ldots, p - 1 \), for any root of multiplicity \( p \). (In particular there is the solution \( e^{rx} \) for a single root).

  - \( x^k e^{ax} \sin bx \) and \( x^k e^{ax} \cos bx \), \( k = 0, \ldots, p - 1 \) for every conjugate complex pair \( a \pm bi \) of roots of multiplicity \( p \). (In particular we get \( e^{ax} \sin bx \) and \( e^{ax} \cos bx \) for a simple conjugate complex pair of roots.)
To solve an IVP with a linear DE we need to do three things:

− Find a particular solution
− Find the general solution of the homogeneous problem.
− Find values of the parameters to match initial conditions:

Example:

\[ y'' - 5y' + 6y = 6x^2 - 10x + 2, \quad y(0) = y'(0) = 1. \]

\[ \begin{align*}
H\Pi: \quad & y'' - 5y' + 6y = 0 \\
C\ E: \quad & r^2 - 5r + 6 = 0 \\
& (r - 2)(r - 3) = 0 \quad r = 2, \ r = 3 \\
& y_c = c_1 e^{2x} + c_2 e^{3x} \\
\end{align*} \]

\[ \begin{align*}
Y_p &= Ax^2 + Bx + C \\
Y'_p &= 2Ax + B \\
Y''_p &= 2A \\
\end{align*} \]

\[ y'' - 5y' + 6y = 6x^2 - 10x + 2 \]

\[ 2A - 5(2Ax + B) + 6(Ax^2 + Bx + C) = 6x^2 - 10x + 2 \]
\[ 6A x^2 + (-10A + 6B)x + 2A - 5B + 6C = Gx^2 - 10x + 2 \]

\[ A = 1 \quad B = 0 \quad C = 0 \]

\[ y(x) = x^2 + C_1 e^{2x} + C_2 e^{3x} \]

\[ y(0) = y'(0) = 1 \]

\[ y'(x) = 2x + 2C_1 e^{2x} + 3C_2 e^{3x} \]

\[ y(0) = C_1 + C_2 = 1 \quad 1.2 \]

\[ y'(0) = 2C_1 + 3C_2 = 1 \]

\[ 2C_1 + 2C_2 = 2 \]

\[ C_2 = -1 \]

\[ C_1 = 2 \]

\[ y(x) = x^2 + 2e^{2x} - e^{3x} \]
\( y'' + y = x \)
\( r^2 + 1 = 0 \quad r = \pm i \)

\[ C_1 \cos x + C_2 \sin x \]

\( y_p(x) = x \)

\[ y(x) = x + c_1 \cos x + c_2 \sin x \]

\[ y(0) = 0 \Rightarrow c_1 = 0 \]

\[ y'(0) = 1 \]

\[ y'(0) = 1 - c_1 \sin 0 + c_2 \cos 0 \]

\[ = 1 \quad c_1 = c_2 = 0 \]

\[ y(0) = y'(0) = 1 \]

\[ 1 = y(0) = c_1 \]

\[ \Rightarrow y(x) = x + \sin x \]