2.4-2.6 Numerical Methods

- Frequently, the initial value problem
  \[ y' = f(x, y), \quad y(a) = y_0 \]
  cannot be solved analytically.

- That does not stop us from using DEs to model the world. If we cannot solve a problem analytically we can still solve it numerically and approximately.

- Example dfield7

- Brief Commercial. The mathematics of answering mathematical questions numerically is called

  **Numerical Analysis**

- We offer many classes: 5600, 5610-20, 6610-20-30

- The simplest method to solve initial value problems is **Euler’s Method**

  \[
  x_n = a + nh, \quad y_n \approx y(x_n), \quad y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, 2, \ldots
  \]

- Leonard Euler, 1707-1783, most prolific mathematician ever.
\[ y_1 = y_0 + hf(x_0, y_0) \]
\[ y_2 = y_1 + hf(x_1, y_1) \]

\[ e_n = y(x_n) - y_n \]
Example:

```maple
> solve y'=y, y(0) = 1; y(1) = e, by Euler's Method
> restart;
> y:=1:
> n:=10;
> h:=1/n:
> for i from 1 to n do y:=y+h*y end do:
> err:=evalf(exp(1)-y);
> err := 0.124539368
> y:=1:
> n:=100;
> h:=1/n:
> for i from 1 to n do y:=y+h*y: end do:
> err:=evalf(exp(1)-y);
> err := 0.013467999
> y:=1:
> n:=1000;
> h:=1/n:
> for i from 1 to n do y:=y+h*y: end do:
> err:=evalf(exp(1)-y);
> err := 0.001357896
> quit
```

“The” Runge-Kutta Method

\[ y_{n+1} = y_n + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4) \]

where

\[ K_1 = f(x_n, y_n) \]
\[ K_2 = f \left( x_n + \frac{h}{2}, y_n + \frac{hK_1}{2} \right) \]
\[ K_3 = f \left( x_n + \frac{h}{2}, y_n + \frac{hK_2}{2} \right) \]
\[ K_4 = f \left( x_n + h, y_n + hK_3 \right) \]
Example:

```plaintext
1 > Digits:=30;

2 > y:=1:
3 > n:=10;
4
5 n := 10

6 > h:=1/n:
7 > for i from 1 to n do
8 > k1:=y;
9 > k2:=y+h/2*k1;
10 > k3:=y+h/2*k2;
11 > k4:=y+h*k3;
12 > y:=y+h/6*(k1+2*k2+2*k3+k4);
13 > end do:
14 > err:=evalf(exp(1)-y);
15
16 err := 0.20843 10

17 > y:=1:
18 > n:=100;
19
20 n := 100

21 > h:=1/n:
22 > for i from 1 to n do
23 > k1:=y;
24 > k2:=y+h/2*k1;
25 > k3:=y+h/2*k2;
26 > k4:=y+h*k3;
27 > y:=y+h/6*(k1+2*k2+2*k3+k4);
28 > end do:
29 > err:=evalf(exp(1)-y);
30
31 err := 0.22464 10

32 > y:=1:
33 > n:=1000;
34
35 n := 1000

36 > h:=1/n:
37 > for i from 1 to n do
38 > k1:=y;
39 > k2:=y+h/2*k1;
40 > k3:=y+h/2*k2;
41 > k4:=y+h*k3;
42
43
```

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\begin{verbatim}
45 > y:=y+h/6*(k1+2*k2+2*k3+k4);
46 > end do:
47 > err:=evalf(exp(1)-y);
48
49 err := 0.22633 10
\end{verbatim}
A Case Study

When my daughter was taking Calculus she asked me how long it would take for the Earth to fall into the Sun if it was to stop dead in its orbit. At the time I was unable to solve the problem analytically and so came up with an answer numerically. My daughter told me that this was not acceptable, she’d like an explicit expression! I have felt challenged by this reaction ever since, and finally, in this note, I am meeting that challenge!

Let’s investigate this question more generally. So suppose we have a star or planet $O$ that has a radius $R$ and a surface gravity $G$ (measured in the appropriate units). For example, the radius of the Sun is $6.96 \times 10^8$ meters, and gravity on the Sun’s surface is 274 meters per second squared. The (average) distance of the Earth from the Sun is $1.49 \times 10^{11}$ meters.

Let’s suppose we release a stationary object $E$ at a distance $H$ from $O$, and we wish to know how long it will take to fall until it reaches a specified distance from $O$. We assume $E$ to be a point, i.e., we ignore its radius. Let $s(t)$ be the distance between $E$ and $O$ at time $t$. $s$ is the solution of the initial value problem

$$s'' = -G \left( \frac{R}{s} \right)^2, \quad s(0) = H, \quad s'(0) = 0. \quad (1)$$

Multiplying with $s'$ on both sides of the differential equation gives

$$s'' s' = -GR^2 \frac{s'}{s^2}. \quad (2)$$

Integrating on both sides gives:

$$\frac{(s')^2}{2} = \frac{GR^2}{s} + C \quad (3)$$

To find the integration constant $C$ we substitute $s = H$
and \( s' = 0 \) to get \( C = -\frac{GR^2}{H} \). Thus

\[
\frac{1}{2}(s')^2 = GR^2 \left( \frac{1}{s} - \frac{1}{H} \right)
\]  

(4)

and

\[
s' = -\sqrt{2GR} \sqrt{\frac{1}{s} - \frac{1}{H}}.
\]  

(5)

The minus sign is due to the fact that the distance from \( O \) is decreasing as time is increasing. The equation (5) is a separable differential equation, giving

\[
dt = -\frac{ds}{\sqrt{2GR} \sqrt{\frac{1}{s} - \frac{1}{H}}},
\]  

(6)

Integrating on both sides gives

\[
t = -\frac{\sqrt{2\sqrt{H} \left( -2\sqrt{s} \sqrt{H} - s + H \arctan \left( \frac{2s-H}{2\sqrt{s} \sqrt{H} - s} \right) \right)}}{4R\sqrt{G}} + T
\]  

(7)

where \( T \) is an integration constant.

This result was first obtained with Maple, and subsequently simplified. You can check by differentiation that it is correct.

Usually one would want \( s \) as a function of \( t \), but actually, for our application, having time expressed as a function of distance is exactly what we need.

We now need to compute the integration constant \( T \). We have \( H \geq s \). Hence the argument of the inverse tangent function in (7) approaches positive infinity as \( s \) approaches \( H \). The inverse tangent itself approaches \( \frac{\pi}{2} \). Taking the limit as \( s \) approaches \( H \) and \( t \) approaches 0 gives

\[
T = \frac{\sqrt{2\pi H^{\frac{3}{2}}}}{8R\sqrt{G}}.
\]  

(8)
Hence the time required for $E$ to reach a distance $s$ from $O$ is

$$t(s) = \frac{\sqrt{2\pi}H^{\frac{3}{2}}}{8R\sqrt{G}} - \frac{\sqrt{2}\sqrt{H}}{4R\sqrt{G}} \left( -2\sqrt{s}\sqrt{H} - s + H \arctan \left( \frac{2s-H}{2\sqrt{s}\sqrt{H-s}} \right) \right)$$

(9)

Substituting the values

$$H = 1.49 \times 10^{11}, \quad R = 0.696 \times 10^{9}, \quad \text{and} \quad s = R$$

(10)

in this expression answers my daughter’s question:

$$t_{\text{impact}} \approx 5,544,225 \text{ seconds} \approx 64.17 \text{ days}.$$

(11)

You might have little confidence in this result since I skipped a few steps. The integral was obtained by a computer program, and I flippantly stated that you can check the answer by differentiation. (I actually did , but, again, I used Maple to carry out that differentiation. That’s not as crazy as it sounds, the differentiation and integration algorithms in Maple are quite distinct.) However, to put your mind at rest, it turns out that the same number, a little more than 64 days, can be obtained numerically.

The method I used is a simple modification of Euler’s method. Let $s_n$ be the approximation of the distance of $E$ from $O$ after $n$ seconds, and let $v_n$ be the approximation of the velocity of $E$ after $n$ seconds. Then the numerical method is as follows:

---

You might also try to obtain the formula (9) via Maple, including the calculation of the integration constant. It’s not as straightforward as it may sound, you will find that Maple insists you are dividing by zero.
\[ s_0 = H, \quad v_0 = 0 \]

\[ \text{For } n = 0, 1, 2, \ldots, \quad \text{until } s_n < R \text{ Do} \]

\[ v_{n+1} = v_n - \frac{GR^2}{s_n^2} h \]

\[ s_{n+1} = s_n + v_{n+1} h \]

(12)

The parameter \( h \) in this outline is the standard notation for the time step, i.e., one second in the actual program. The method makes physical sense: we update the velocity, and then we use the new velocity to update the distance. The velocity is negative, and getting more so, since the positive direction is up, away from \( O \).

The following very simple Fortran program actually runs the method:

```fortran
1 implicit double precision (a-z)
2 Earth = 1.49E11
3 Sun = 0.696E9
4 Gravity = 274
5 time = 0
6 step = 1
7 distance = Earth
8 speed = 0
9 continue
10 time = time + step
11 speed = speed - Gravity*Sun*Sun/distance/distance*step
12 distance = distance + speed*step
13 if (distance > Sun) go to 100
14 write(*,*) time, speed, distance
15 write(*,*) time/86400
16 stop
17 end
```

It’s output is

```
5544225.  -616269.809  695396141.
64.1692708
```

The last number is the impact time measured in days, which equals the time measured in seconds and divided by
86,400, the number of seconds in one day. The numerical results are consistent with the analytical results.

For the fun of it, and since we’ve spent all that work, let’s compute how long it would take for the moon to fall onto the Earth. The radius of the Earth is 6,378,000 meters, gravity on the surface of the Earth is 9.8 meters per second squared, and the distance of the moon from the Earth is 384 million meters. Substituting these values into our formula gives

$$t_{\text{impact}} \approx 418,222 \text{ seconds} \approx 4.8 \text{ days} . \quad (13)$$

Again, these numbers are consistent with the computational results.

**Acknowledgment:** The breakthrough in this enterprise occurred in a conversation with my colleague Grant Gustafson who encouraged me to use, and trust, Maple.
Maple Code

restart;
Z := 1/sqrt(2*G)/R/sqrt(1/s-1/E);
t := int(Z, s);
tt := -(sqrt(2)/4*sqrt(E)*(-2*sqrt(s)*sqrt(E-s)+E*arctan((2*s-E)/2/sqrt(s)/sqrt(E-s)))/sqrt(G)/R
+ sqrt(2)*Pi*E^(3/2)/8/R/sqrt(G));
zero := simplify(diff(tt, s)-Z);
assume(s>0);
assume(E>0);
assume(R>0);
t := limit(t, s=E, 'left')-t;
travel := subs(G = gravity, R = Sun, E = Earth, s=distance,t);
T := subs(distance=Sun,travel);
gravity := 274;
Earth := 1.49E11;
Sun := 0.696E9;
impact := evalf(T/86400);
plot(travel,distance = Earth..Sun);
\begin{verbatim}
\textbf{Maple Calculation}

Maple 2016 (X86 64 LINUX)
\textcopyright Maplesoft, a division of Waterloo Maple Inc. 2016
\textcopyright Waterloo Maple Inc.

\begin{verbatim}
| Type ? for help.
\end{verbatim}

\begin{verbatim}
> restart;
>
> Z:=1/sqrt(2*G)/R/sqrt(1/s-1/E);

Z := -----------------
    1/2 1/2
    2 G R (1/s - 1/E)

> t:=int(Z,s);

2 1/2  2 s
\ |----|\ s E \ E arctan(----
  1/2 1/2 \ | 2 (E s - s ) | \ s E / | 2

2
\ - s ) / 2 (E s

\textbf{t} := -1/4 \---------------------
\textbf{---------------------}
\begin{verbatim}
1/2 1/2
G R (s (E - s))
\end{verbatim}

\begin{verbatim}
> tt:=-(sqrt(2)/4*sqrt(E)*(-2*sqrt(s)*sqrt(E-s)+E*arctan((2*s-E)/2/sqrt(s)/sqrt(E-s)))/sqrt(G)/R

> + sqrt(2)*Pi*E^(-3/2)/8/R/sqrt(G));

2 1/2  1/2 / 1/2 1/2
1/2 s - E \ E

2 1/2
\ -2 s (E - s) + E arctan(-
\textbf{tt} :=-----------------------------------)

\end{verbatim}
\end{verbatim}

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\end{verbatim}
\[ tt := -\frac{1}{4} \frac{1/2}{G R} \]

\[ \frac{1/2}{2 \ \text{Pi} E} \]

\[ \frac{3/2}{8 \text{R} G} \]

\[
> \text{zero:=simplify(diff(tt,s)-Z);} \\
> 1/2 \sqrt{E - s} \frac{1}{2} + s (E - s) \frac{1}{2} \\
\]

\[
\frac{1/2}{2} \sqrt{E - s} \frac{1}{2} R G (E - s) s \]

\[
\text{assume}(s>0); \text{assume}(E>0); \text{assume}(R>0); \]

\[
> t:=\text{limit}(t,s=E, 'left')-t; \\
3/2 \sqrt{E^2 - 2 s^2} \frac{1}{2} (E^2 - s^2) \frac{1}{2} \\
\frac{1/2}{2} \sqrt{E - s^2} \frac{1}{2} \text{arctan}\left(\frac{E^2 - 2 s^2}{(G R)^2}\right) + 2 (E^2 s^2 - s^2) \frac{1}{2} \\
\text{Math 2280-1 Notes of June 2, 2016 page 14}
> travel := subs(G = gravity, R = Sun, E = Earth, s = distance, t);
   3/2 1/2
   Earth Pi 2 1/2 /Earth - distance\1/2 /
   travel := ---------------- + 1/4 2 |--------
   Earth / | distance Earth |
   1/2 \ distance

> Earth arctan(-------------------------------
   --)
   2 1/2
   2 (Earth distance - distance )

> + 2 (Earth distance - distance ) | / (gravity Sun
   1/2 /

> (distance (Earth - distance)) )

> T := subs(distance = Sun, travel);
   3/2 1/2
   Earth Pi 2 1/2 /Earth - Sun\1/2 /
   T := ---------------- + 1/4 2 |--------| Earth
   1/2 \ Sun Earth /

> 8 gravity Sun

> Earth - 2 Sun

> gravity := 274;
gravity := 274

> Earth:=1.49E11;
Earth := 0.149 10

> Sun:=0.696E9;
Sun := 0.696 10

> impact:=evalf(T/86400);
impact := 64.16926827

> plot(travel,distance = Earth..Sun);
155 > quit
156